Math 181 – Final Exam Info and Review Exercises

Fall 2018, Prof. Beydler

Test Info

- Date: Monday, December 10, 2018 from 7:30pm-10:00pm
- Will cover almost all sections.
- This will be a 2-part test. Part 1 will be **no calculator**. Part 2 will be **scientific calculator only**.
- No notes, no books, no phones, no smart watches during the final exam. Please don't fail the class because of a phone in your lap!
- As usual, there will be a seating chart for the final exam.
- Where to get help as you're studying:
  - Office hours
  - TMARC, LAC, or other tutoring centers
  - E-mail me at dbeydler@mtsac.edu
- If you go a Mt. SAC tutoring center for 4 hours between Test #3 and the Final Exam, you'll get 1% extra credit towards the Final Exam.

Not on the test

Trapezoidal and Simpson's Rule (7.7)

Remainder Estimate for the Integral Test and the infamous $\boxed{3}$ formula (part of 11.3)

Formulas and stuff

(Note: This list is not meant to include everything you'll need to know on the test.)

Cross-sections

$V = \int_a^b A(x)\,dx$

Shell method

$V = \int_a^b 2\pi(\text{radius})(\text{height})\,dx$

Work

$W = \int_a^b F(x)\,dx$

Spring force

$F = kx$

Integration by parts

$\int u\,dv = uv - \int v\,du$

Trig formulas

$\sin^2 x + \cos^2 x = 1$

$\sin^2 x = \frac{1-\cos 2x}{2}$

$\cos^2 x = \frac{1+\cos 2x}{2}$

$\tan^2 x + 1 = \sec^2 x$

$1 + \cot^2 x = \csc^2 x$

Trig substitutions

$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$0 \le \theta < \frac{\pi}{2}$ (for $a > 0, x > 0$)

Partial fraction decomposition

Linear factor $(x-r)$: $\frac{A}{x-r}$

Repeated linear factor $(x-r)^m$: $\frac{A_1}{x-r}+\frac{A_2}{(x-r)^2}+\cdots+\frac{A_m}{(x-r)^m}$

Irreducible quadratic factor x^2+px+q: $\frac{Ax+B}{x^2+px+q}$

Repeated irreducible quadratic factor $(x^2+px+q)^n$: $\frac{B_1x+C_1}{(x^2+px+q)}+\frac{B_2x+C_2}{(x^2+px+q)^2}+\cdots+\frac{B_nx+C_n}{(x^2+px+q)^n}$

Improper integrals (Type I)

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$$

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$$

Improper integrals (Type II)

$$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx \qquad \text{(discontinuity at } x=a\text{)}$$

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx \qquad \text{(discontinuity at } x=b\text{)}$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \qquad \text{(discontinuity at } x=c\text{)}$$

$\frac{1}{x^p}$ integrals

$\int_1^\infty \frac{1}{x^p}\,dx$ converges if $p>1$ and diverges if $p\le 1$.

Comparison Test

Suppose f and g are continuous on $[a,\infty)$ and $0\le f(x)\le g(x)$ in $[a,\infty)$.

If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges.

If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ diverges.

Arc length

$L=\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ (or $L=\int_a^b \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$)

Surface area

$S=\int 2\pi r\,ds$ (often $S=\int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ or $S=\int_a^b 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$)

Hydrostatic force

$P_i = \rho g d,\ F_i = P_i A_i$

Centroid (center of mass with uniform density)

$\overline{x} = \frac{1}{A}\int \tilde{x}\, dA \qquad \overline{y} = \frac{1}{A}\int \tilde{y}\, dA \quad (\tilde{x}, \tilde{y})$is the centroid of a thin strip, and $dA = (\text{length of strip}) \cdot dx$

Parametric Curves

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$

$$\text{Net area} = \int y\, dx = \int_{t=a}^{t=b} g(t) f'(t) dt$$

$$\text{Arc length} = \int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\text{Surface area} = \int_{t=a}^{t=b} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad \text{or} \qquad \int_{t=a}^{t=b} 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Polar Coordinates

$x = r\cos\theta\,,\ \ y = r\sin\theta \qquad\qquad r^2 = x^2 + y^2,\ \ \tan\theta = \frac{y}{x}$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Horizontal tangent lines will happen when $\frac{dy}{d\theta} = 0$ and $\frac{dx}{d\theta} \neq 0$.

Vertical tangent lines will happen when $\frac{dx}{d\theta} = 0$ and $\frac{dy}{d\theta} \neq 0$.

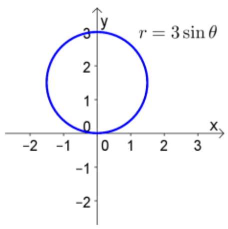
If both $\frac{dx}{d\theta} = 0$ and $\frac{dy}{d\theta} = 0$ at $\theta = \theta_0$, then you'll have to check $\lim_{\theta \to \theta_0} \frac{dy}{dx}$ and possibly use L'Hospital.

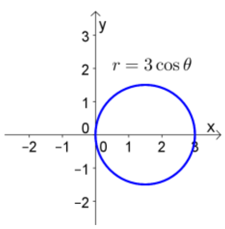
$$\text{Area} = \int_{\theta=\alpha}^{\theta=\beta} \frac{1}{2} r^2 d\theta$$

$$\text{Arc length} = \int_{\theta=\alpha}^{\theta=\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

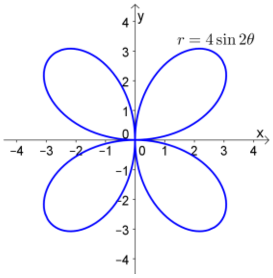
Know these polar graphs well!!

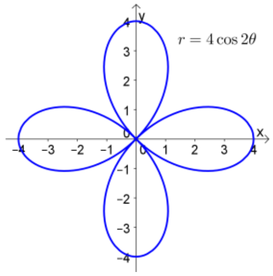
Circle: $r = a\sin\theta$ and $r = a\cos\theta$

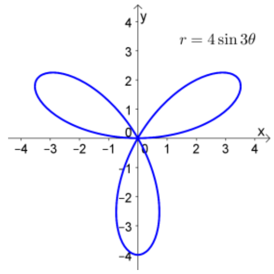




Rose: $r = a\sin n\theta$ and $r = a\cos n\theta$

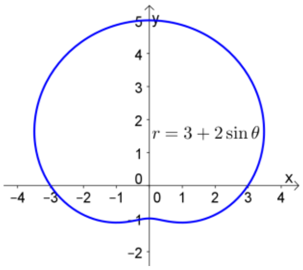


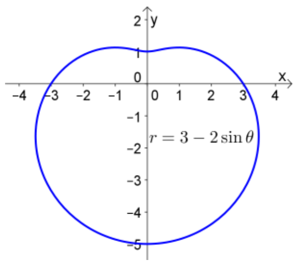




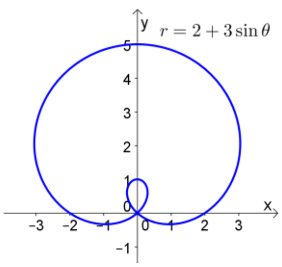
Limacon: $r = a \pm b\sin\theta$ and $r = a \pm b\cos\theta$

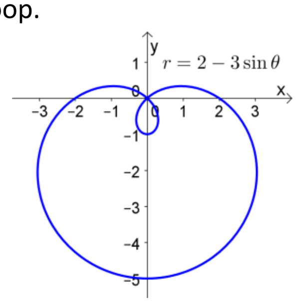
If $a > b$, there is no inner loop.



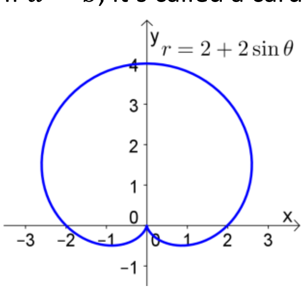


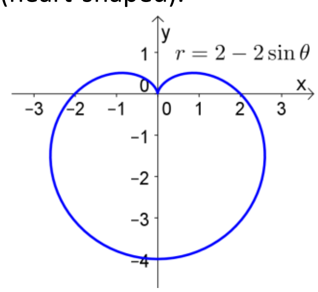
If $a < b$, there is an inner loop.





If $a = b$, it's called a cardioid (heart-shaped).





Series

Geometric series: $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$ converges to $\frac{a}{1-r}$ if $|r| < 1$, diverges if $|r| \geq 1$

p-series: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$, and diverges if $p \leq 1$

Does a series converge or diverges? Here are the tests we've learned so far…

Test for Divergence: If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

The Integral Test: Suppose that $a_n = f(n)$, where $f(x)$ is **continuous**, **positive**, and **decreasing** for all $x \geq N$.
Then $\sum_{n=N}^{\infty} a_n$ and $\int_N^{\infty} f(x)\,dx$ both converge or both diverge.

The Comparison Test: Suppose a_n and b_n have nonnegative terms, and N is some integer.
If $a_n \leq b_n$ for all $n > N$ and if $\sum b_n$ converges, then the smaller $\sum a_n$ also converges.
If $b_n \leq a_n$ for all $n > N$ and if $\sum b_n$ diverges, then the bigger $\sum a_n$ also diverges.

The Limit Comparison Test: Suppose a_n and b_n have positive terms for all $n \geq N$ (N is some integer).

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Alternating Series Test (AST): $\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots$ converges if:

1. $b_n > 0$
2. $b_{n+1} \leq b_n$ for all $n \geq N$
3. $b_n \to 0$

The Absolute Convergence Test

If $\sum |a_n|$ converges, then $\sum a_n$ converges.

The Ratio Test

Suppose that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$.
If $L < 1$, then $\sum a_n$ converges absolutely.
If $L > 1$ (or L infinite), then $\sum a_n$ diverges.
If $L = 1$, then test inconclusive (try something else).

The Root Test

Suppose that $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$
If $L < 1$, then $\sum a_n$ converges absolutely.
If $L > 1$ (or L infinite), then $\sum a_n$ diverges.
If $L = 1$, then test inconclusive (try something else).

Maclaurin series of f: $\boldsymbol{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n} = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$

Taylor series of f at a: $\boldsymbol{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n} = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$

How well does a partial sum approximate the infinite sum?

Remainder Estimate for the Integral Test: Suppose that $a_n = f(n)$, where $f(x)$ is continuous, positive, and decreasing for all $x \geq n$. If $\sum a_n$ converges, then $\boldsymbol{\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx}$

Alternating Series Estimation Theorem

Suppose we have an alternating series $s = \sum_{n=1}^{\infty}(-1)^{n-1}b_n$ where $b_n > 0$, $b_{n+1} \leq b_n$, and $b_n \to 0$.
Then, $|R_n| = |s - s_n| \leq b_{n+1}$

I'll give you these formulas if you need them:

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

$$\int \csc x\,dx = -\ln|\csc x + \cot x| + C$$

Common Taylor series:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \qquad |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \qquad -1 < x \leq 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \qquad |x| \leq 1$$

$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$ where $\binom{k}{n} = \frac{k(k-1)(k-2)\ldots(k-n+1)}{n!}$ and interval of convergence is $-1 < x < 1$

Review Exercises

Note: If you write up the answers to all of the review exercises listed below, and hand them in at the final exam, you can earn up to 2% extra credit towards your final exam! It is important to understand that these review exercises are not guaranteed to cover all of the potential problems on the final exam. Please review the notes, homework problems, previous tests, and previous test review exercises to fully prepare for the final exam. Types of problems that will appear on Part 1 are labeled **NC** (for **N**o **C**alculator).

1. Find the area of the region(s) enclosed by the curves $y = \sin^2 x$ and $y = \cos^2 x$ from $x = \frac{\pi}{4}$ to $x = \frac{5\pi}{4}$.

2. A solid lies between the planes perpendicular to the x-axis at $x = 0$ and $x = 2$. The cross-sections perpendicular to the x-axis between these planes are squares whose diagonals run from the curve $y = x^2$ to the curve $y = 2x$. Find the volume of the solid.

3. Find the volume of the solid generated by revolving the region bounded by the following curves about the line $y = -1$.
$y = 2\sqrt{x},\ \ y = 2,\ x = 0$

4. Find the volume of the solid generated by revolving the region bounded by the following curves about the line $y = 1$.
$x = y - y^3,\ x = 0,\ y \geq 0$

5. A force of 10 lb is required to hold a spring stretched 1 ft beyond its natural length. How much work is done in stretching it from its natural length to 4 ft beyond its natural length?

6. A 15-kg rope is 5-m long and hangs down from a 20-m-tall building. A 3-kg Calculus book is tied to the end of the rope. Assuming the rope's mass is uniformly distributed, how much work is done lifting the rope to the top of the building?

7. An inverted conical tank (tip down) has a height of 10 ft and a radius of 5 ft. It is filled to within 2 ft of the top with water. How much work is required to pump the water to a distance of 3 ft above the top of the tank?

8. Find the average value of $f(x) = x^2 \sin x$ on the interval $\left[0, \frac{\pi}{2}\right]$.

9. Find the following integrals. If the integral diverges, write "diverges." (**NC**)

a. $\int_0^1 \cos^{-1} x \, dx$

b. $\int \tan^2 x \sec^4 x \, dx$

c. $\int \frac{dx}{(4-x^2)^{3/2}}$

d. $\int \frac{x^3 dx}{\sqrt{x^2+4}}$

e. $\int \frac{9x^3-3x+1}{x^3-x^2}dx$

f. $\int \frac{-2x+4}{(x^2+1)(x-1)^2}dx$

g. $\int_{-1}^{2} \frac{1}{x^2} dx$

h. $\int_{-\infty}^{\infty} 2xe^{-x^2}\, dx$

i. $\int e^{\sqrt[3]{x}}\,dx$

j. $\int \frac{1}{x\sqrt{x^2+1}}\,dx$

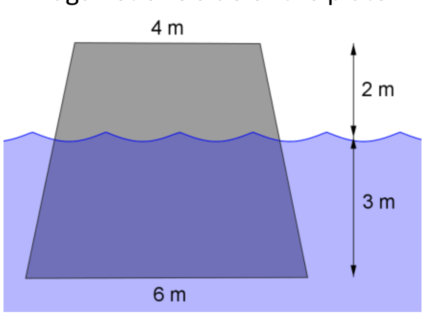
k. $\int \frac{1}{1+e^x} dx$

10. Show whether $\int_2^{\infty} \frac{x+3}{\sqrt{4x^4-3x-1}} dx$ converges or diverges using the Comparison Test. (**NC**)

11. Find the length of $y = \frac{x^6+8}{16x^2}$ for $x = 2$ to $x = 3$.

12. Find the area of the surface generated by revolving the curve $y = \sqrt{4 - x^2}$ from $x = -1$ to $x = 1$ about the x-axis.

13. A vertical plate in the shape of a trapezoid is partially submerged in water as shown. Use a Riemann sum to approximate the hydrostatic force against one side of the plate. Then find the exact hydrostatic force against one side of the plate.



14. Find the centroid of the region enclosed by $y = x^2 - x$ and $y = x$.

15. Show that $y = t\sin t$ is a solution to the following initial-value problem. (Note: Here, you have to show that the DE and **both** initial conditions are satisfied.) (**NC**)

$y'' + 3y' + y = 3t\cos t + 3\sin t + 2\cos t, \quad y(0) = 0, \;\; y'(0) = 0$

16. Consider the following DE: $\frac{dy}{dt} = (y-2)(y-1)$. (**NC**)

a. For what values of y will $y(t)$ be decreasing?

b. For what values of y will $y(t)$ be increasing?

c. For what value(s) of y will $y(t)$ be constant?

17. Solve the following initial value problems. Be sure to explicitly solve for y.

a. $e^x y' = xy,\ y(0) = 1$

b. $y' = y\sin x,\ y(0) = \frac{e}{2}$

18. Find the orthogonal trajectories of the family of curves $kx^2 + y^2 = 1$, where k is an arbitrary constant. (**NC**)

19. A 100-gallon tank is full of a solution that's 6% alcohol. A solution with 10% alcohol starts pouring into the tank at 20 gal/min. Assume the solution in the tank is kept well-mixed and drains from the tank at 20 gal/min. Find the percentage of alcohol in the tank's solution after 20 minutes.

20. Find an equation for the line tangent to the curve $x = t^2,\ y = \tan t$ at the point where $t = \frac{\pi}{4}$.

21. Find the length of the curve $x = t,\ y = e^t,\ 0 \leq t \leq 1$

22. Find the area of the region…
    a. …enclosed by $r = 4\cos\theta$.

    b. …inside $r = 1 + \sin\theta$ and outside $r = 3\sin\theta$.

c. …inside one leaf of $r = \cos 3\theta$.

23. Find the length of the curve $r = \sin^2 \frac{\theta}{2}$, $0 \leq \theta \leq \pi$.

24. Find the slope of the curve $r = 2 + \cos 2\theta$ at $\theta = \frac{\pi}{3}$.

25. Find the values of θ in $[0,2\pi)$ where the tangent line of $r = e^{\theta}$ is horizontal or vertical.

26. Determine whether each sequence converges or diverges. If it converges, find the limit. (**NC**)

a. $a_n = e^{2\cos^{-1}\left(\frac{1}{n}\right)}$

b. $a_n = \frac{2n^2}{\sqrt{n^3+3n-1}}$

c. $a_n = \frac{(\ln n)^2}{\sqrt[3]{n}}$

d. $a_n = \frac{3^{n+1}+2^{3n}}{5^{n+2}}$

27. Assume that the following sequence converges and find its limit.

$a_1 = 1, \quad a_{n+1} = \frac{a_n+3}{3}$

28. Determine whether each series is convergent or divergent. Be sure to state any test that you use and show your reasoning. If you use the Integral Test or Alternating Series Test, be sure to state the conditions of the test and (if necessary) show why the conditions are met. (**NC**)

a. $\sum_{n=2}^{\infty} e^{-2n}$ (If it converges, what does it converge to?)

b. $\sum_{n=1}^{\infty}(-1)^n \ln\left(\frac{n+2}{n+1}\right)$

c. $\sum_{n=2}^{\infty}\frac{1}{n\sqrt{\ln n}}$

d. $\sum_{n=1}^{\infty}\frac{(-2)^{3n}}{7^{n+1}}$ (If it converges, what does it converge to?)

e. $\sum_{n=1}^{\infty}\frac{2}{(n+1)(n+3)}$ (If it converges, what does it converge to?)

f. $\sum_{n=1}^{\infty}\left(\frac{\sqrt{n^2+1}}{2n+1}\right)^{2n}$

g. $\sum_{n=1}^{\infty}\frac{1}{2\sqrt{n}+\sqrt[3]{n}}$

h. $\sum_{n=1}^{\infty}\frac{2\sin^2 n}{n^{3/2}}$

29. Determine whether each series converges absolutely, converges conditionally, or diverges. (**NC**)

a. $\sum_{n=1}^{\infty}\frac{n2^n(n+1)!}{3^n n!}$

b. $\sum_{n=1}^{\infty} \frac{(\ln n)^n}{n^n}$

c. $\sum_{n=0}^{\infty} (-1)^n \frac{3}{n+1}$

d. $\sum_{n=1}^{\infty} \frac{(-2)^{n+1}}{n+5^n}$

30. Find the radius and interval of convergence of each series. (**NC**)

a. $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x+1)^n}{n+2}$

b. $\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$

31. Express $\frac{x^2}{x-2}$ as a power series and find its interval of convergence. (**NC**)

32. Find the Maclaurin series for $\ln(4 - x)$ and find its radius of convergence.

33. Find the Taylor series for 2^x centered at $x = 1$ using the definition of a Taylor series. Also, find its radius of convergence.

34. Find the Taylor series for $\sqrt{x}$ centered at $x = 1$ using the definition of a Taylor series. Also, find its radius of convergence.

35. Find the Maclaurin series for xe^{2x^2} and find its radius of convergence. (**NC**)

36. Find the Maclaurin series for $\frac{1}{\sqrt[3]{8-x}}$ and find its radius of convergence. (**NC**)

37. Find an infinite series representation for $\int x^2 \cos x^2 \, dx$. (**NC**)

38. Evaluate $\int_0^{0.2} x^2 e^{-x^2} \, dx$ correct to within an error of 0.001.

39. Evaluate $\lim_{x \to \infty} x^2\left(e^{-1/x^2} - 1\right)$ using series. (**NC**)

40. Find the first three nonzero terms in the Maclaurin series for $(\tan^{-1} x)^2$.

41. Find the first three nonzero terms in the Maclaurin series for $\frac{\ln(1+x)}{1-x}$.

42. Find the Taylor polynomial $T_3(x)$ for $f(x) = x \cos x$ centered at $a = 0$.

43. Find the Taylor polynomial $T_3(x)$ for $f(x) = x \ln x$ centered at $a = 1$. Then use Taylor's Inequality to estimate the accuracy of $T_3(x)$ when x lies in the interval $0.5 \le x \le 1.5$.