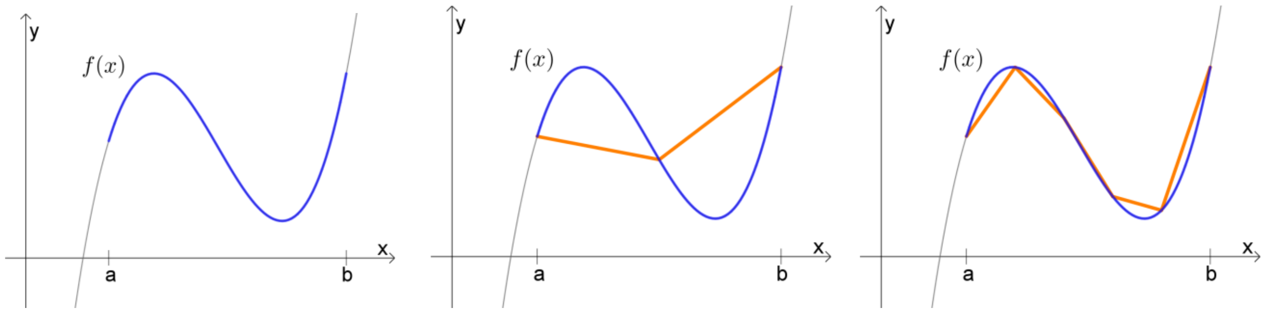
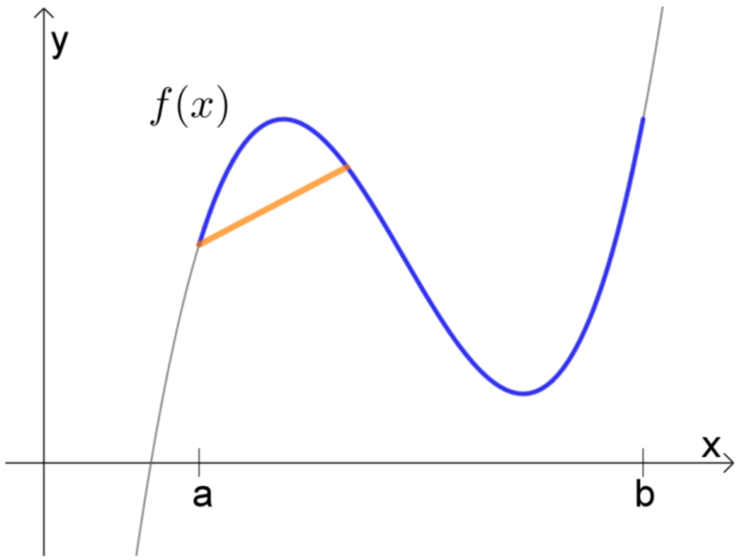
Arc Length





So, we have $\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2}$. Factor out a $(\Delta x)^2$ to get $\Delta s = \sqrt{(\Delta x)^2\left(1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right)}$.

By the Mean Value Theorem, there must be a x_i^* in the subinterval such that $\frac{\Delta y}{\Delta x} = f'(x_i^*)$.

So, $\Delta s = \sqrt{1 + [f'(x_i^*)]^2}\Delta x$.

Now, the approximate arc length is $\sum_{i=1}^{n} \Delta s = \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\Delta x$.

Taking the limit as $n \to \infty$, we get $\int_a^b ds = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$. This integral finds the arc length!

Here's a more formal statement of the arc length formula…

If f' is continuous on $[a, b]$, then the **length** (**arc length**) of the curve $y = f(x)$ from $(a, f(a))$ to $(b, f(b))$ is:

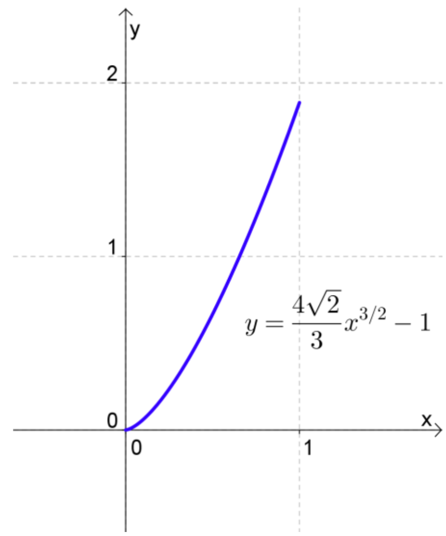
$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

$$= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

(Note: the "f' is continuous on $[a, b]$" part says that the curve must be **smooth**, meaning it can't have any breaks, corners, or cusps.)

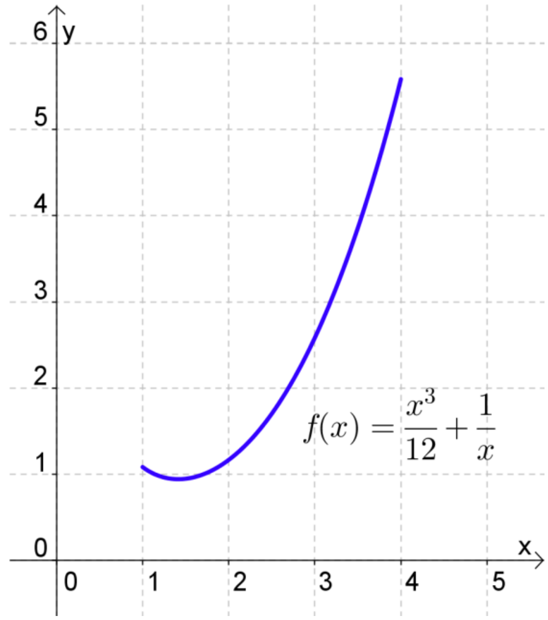
Ex 1.

Find the length of the curve $y = \frac{4\sqrt{2}}{3}x^{3/2} - 1, \quad 0 \le x \le 1$



Ex 2.

Find the length of the curve $f(x) = \frac{x^3}{12} + \frac{1}{x}$, $\quad 1 \le x \le 4$



Note:

When we have x as a function of y, we can write the arc length integral as:

$$L = \int_a^b \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$