Integration of Rational Functions by Partial Fractions

Let's review the **method of partial fractions**, which is a technique to break rational functions up into the sum of simpler rational functions.

Example: $\frac{5x-3}{x^2-2x-3} = \frac{5x-3}{(x+1)(x-3)} = \frac{2}{x+1} + \frac{3}{x-3}$

Ex 1.

Expand the quotient by partial fractions.

$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)}$

Ex 2.

Evaluate the following integral.

$\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx$

In general, we need to factor the denominator first. What happens if we don't get linear factors?

1. If we get a ***repeated*** **linear factor** $(\boldsymbol{x-r})^{\boldsymbol{m}}$, then we'll have corresponding partial fractions: $\frac{\boldsymbol{A_1}}{\boldsymbol{x-r}} + \frac{\boldsymbol{A_2}}{(\boldsymbol{x-r})^2} + \cdots + \frac{\boldsymbol{A_m}}{(\boldsymbol{x-r})^{\boldsymbol{m}}}$
2. If we get an **irreducible quadratic factor** $\boldsymbol{x^2+px+q}$, then we'll have a corresponding partial fraction: $\frac{\boldsymbol{Ax+B}}{\boldsymbol{x^2+px+q}}$
3. If we get a ***repeated*** **irreducible quadratic factor** $(\boldsymbol{x^2+px+q})^{\boldsymbol{n}}$, then we'll have corresponding partial fractions: $\frac{\boldsymbol{B_1x+C_1}}{(\boldsymbol{x^2+px+q})} + \frac{\boldsymbol{B_2x+C_2}}{(\boldsymbol{x^2+px+q})^2} + \cdots + \frac{\boldsymbol{B_nx+C_n}}{(\boldsymbol{x^2+px+q})^{\boldsymbol{n}}}$

Ex 3.

$\int \frac{6x+7}{(x+2)^2}\,dx$

Note: If degree of top polynomial is ___ degree of bottom polynomial, then ________ first.

Ex 4.

$\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}dx$

Ex 5.

$\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx$

If your integral has $\sqrt{\text{something}}$, you might try letting $u = \sqrt{\text{something}}$.

Ex 6.

$\int \frac{\sqrt{x+1}}{x}\,dx$

Practice

1. Evaluate each integral.

   a) $\int \frac{x^2+8}{x^2-5x+6}\,dx$

b) $\int \frac{1}{x(x^2+1)^2}\,dx$ (Hint: For part b, it's probably easier to expand and match coefficients rather than plug in x-values. This often helps with irreducible quadratic factors.)

Q: What do you get when you expand $(x-a)(x-b)(x-c)\ldots(x-y)(x-z)$?