Absolute Convergence and the Ratio and Root Tests

$\sum a_n$ ________________________________ if $\sum|a_n|$ converges.

The Absolute Convergence Test
If $\sum|a_n|$ converges, then $\sum a_n$ converges.

Ex 1.
Does $\sum_{n=0}^{\infty}(-1)^n \frac{1}{2^n}$ converge or diverge?

Ex 2.
Does $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converge or diverge?

Note: If $\sum a_n$ converges, must $\sum|a_n|$ converge? ________
For example, $\sum(-1)^n \frac{1}{n}$ converges, but $\sum \frac{1}{n}$ diverges.

$\sum a_n$ ________________________________ if $\sum a_n$ converges and $\sum|a_n|$ diverges.

Ex 3.
For what values of $p > 0$ does $\sum \frac{(-1)^{n-1}}{n^p}$ converge conditionally and converge absolutely?

Remember how geometric series $\sum ar^{n-1}$ converges when $|r| < 1$?
In general for a series with positive terms, if (as $n \to \infty$) the *next* term is a *fraction* (< 1) of the *previous* term, then the series will converge. This is what the Ratio Test says.

The Ratio Test
Suppose that
$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$$
If $L < 1$, then $\sum a_n$ converges absolutely.
If $L > 1$ (or L infinite), then $\sum a_n$ diverges.
If $L = 1$, then test inconclusive (try something else).

Ex 4.
Does $\sum_{n=0}^{\infty} \frac{2^n+5}{3^n}$ converge or diverge?

Ex 5.
Does $\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$ converge or diverge?

Ex 6.

Does $\sum_{n=1}^{\infty} \frac{4^n n! n!}{(2n)!}$ converge or diverge?

Suppose you rewrite a series $\sum a_n$ as $\sum\left(\sqrt[n]{a_n}\right)^n$ and suppose that $\sqrt[n]{a_n} \to \frac{2}{3}$. Then $\sum\left(\sqrt[n]{a_n}\right)^n$ acts more and more like a geometric series ($r \approx \frac{2}{3}$) as n gets large, which converges. This is the idea of the Root Test.

The Root Test

Suppose that $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$

If $L < 1$, then $\sum a_n$ converges absolutely.

If $L > 1$ (or L infinite), then $\sum a_n$ diverges.

If $L = 1$, then test inconclusive (try something else).

Ex 7.

Does $\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$ converge or diverge?

Ex 8.

Does $\sum_{n=1}^{\infty}\left(\ln\left(e^2-\frac{1}{n}\right)\right)^{n+1}$ converge or diverge?

Notes:

- The Ratio Test is typically good with factorials.
- The Root Test works best when the terms are raised to the nth power.

Practice

1. Does $\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1}{n\ln n}$ converge absolutely, converge conditionally, or diverge?

2. Does $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(n!)^2}{(2n)!}$ converge absolutely, converge conditionally, or diverge?

3. Does $\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+2)^2}$ converge or diverge?

4. Does $\sum_{n=1}^{\infty} \frac{4^n}{(3n)^n}$ converge or diverge?

5. Does $\sum_{n=1}^{\infty} a_n$ converge or diverge, if $\{a_n\}$ is defined by $a_1 = 1,\ a_{n+1} = \frac{1+\tan^{-1} n}{n} a_n$? (Hint: the Ratio Test is good with recursively-defined terms.)

6. Does $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ converge or diverge?

7. Does $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$ converge or diverge?

Challenge: If $a_n = \begin{cases} n/2^n & \text{if } n \text{ is a prime number} \\ 1/2^n & \text{otherwise} \end{cases}$, does $\sum_{n=1}^{\infty} a_n$ converge or diverge?

Q: What goes up and down but doesn't move?