1. Does $\sum_{n=1}^{\infty}(-1)^n \frac{1}{\sqrt[7]{n}}$ converge or diverge?

$b_n = \frac{1}{\sqrt[7]{n}}$

① $\frac{1}{\sqrt[7]{n}} > 0$ for $n \geq 1$

② $\frac{1}{\sqrt[7]{n+1}} \leq \frac{1}{\sqrt[7]{n}}$ for $n \geq 1$

③ $\frac{1}{\sqrt[7]{n}} \to 0$

So by AST, $\sum_{n=1}^{\infty}(-1)^n \frac{1}{\sqrt[7]{n}}$ $\boxed{\text{converges}}$.

2. Does $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{n^2}{n^3+1}$ converge or diverge?

$b_n = \frac{n^2}{n^3+1}$

① $\frac{n^2}{n^3+1} > 0$ for $n \geq 1$

② $\frac{d}{dn}\left(\frac{n^2}{n^3+1}\right) = \frac{(n^3+1)(2n)-(n^2)(3n^2)}{(n^3+1)^2} = \frac{2n^4+2n-3n^4}{(n^3+1)^2} = \frac{n(2-n^3)}{(n^3+1)^2}$

negative when
$2-n^3<0$
$n^3>2$
$n>\sqrt[3]{2}$

So, $b_{n+1} \leq b_n$ for $n \geq 2$

③ $\lim_{n\to\infty} \frac{n^2}{n^3+1} = \lim_{n\to\infty} \frac{1/n}{1+1/n^3} = 0$

So by AST, $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{n^2}{n^3+1}$ $\boxed{\text{converges}}$.

Challenge: Show by example that $\sum a_n b_n$ may diverge even if $\sum a_n$ and $\sum b_n$ both converge.

$\sum \frac{(-1)^n}{\sqrt{n}}$ and $\sum \frac{(-1)^n}{\sqrt{n}}$ both converge,

but $\sum \frac{(-1)^n}{\sqrt{n}} \cdot \frac{(-1)^n}{\sqrt{n}} = \sum \frac{1}{n}$ diverges.

Q: What is harder to catch the faster you run?