**1.** Does $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ converge or diverge? (Use the Integral Test. Be sure to show why the terms are decreasing.)

$f(x) = \frac{x}{x^2+1}$ is continuous, positive, and decreasing for $x \geq 1$.

Why decreasing? $f'(x) = \frac{(x^2+1)(1) - x \cdot 2x}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$ is negative for $x > 1$.

$$\int_1^{\infty} \frac{x}{x^2+1}\,dx = \lim_{t\to\infty} \int_1^t \frac{x}{x^2+1}\,dx$$

$u = x^2+1$
$du = 2x\,dx$
$\frac{1}{2}du = x\,dx$
$x=1:\ u=2$
$x=t:\ u=t^2+1$

$$= \lim_{t\to\infty} \frac{1}{2}\int_2^{t^2+1} \frac{1}{u}\,du$$

$$= \lim_{t\to\infty} \frac{1}{2}\left[\ln u\right]_2^{t^2+1}$$

$$= \lim_{t\to\infty} \frac{1}{2}\left[\ln(t^2+1) - \ln 2\right]$$

$$= \infty$$

So, by Integral Test, since $\int_1^{\infty} \frac{x}{x^2+1}\,dx$ diverges, $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ also $\boxed{\text{diverges}}$.

**Challenge:** Is there a "smallest" divergent series? (That is, a divergent series $\sum a_n$ where the terms $a_n$ are smaller than the terms of any other divergent series.)

$\boxed{\text{No.}}$

From any divergent series $\sum a_n$, we can make a divergent series with smaller terms – just divide each term by 2: $\sum \frac{a_n}{2}$

Why does $\sum \frac{a_n}{2}$ diverge? Suppose $\sum \frac{a_n}{2}$ converged. (✱)

Then $2\sum \frac{a_n}{2} = \sum 2 \cdot \frac{a_n}{2} = \sum a_n$ converges.

But this contradicts the fact that $\sum a_n$ diverges, so (✱) must be false, and $\sum \frac{a_n}{2}$ must diverge. ▨

Q: A man leaves home and, after making three left turns, he ends up back at home, and finds two masked men waiting for him. What is happening?