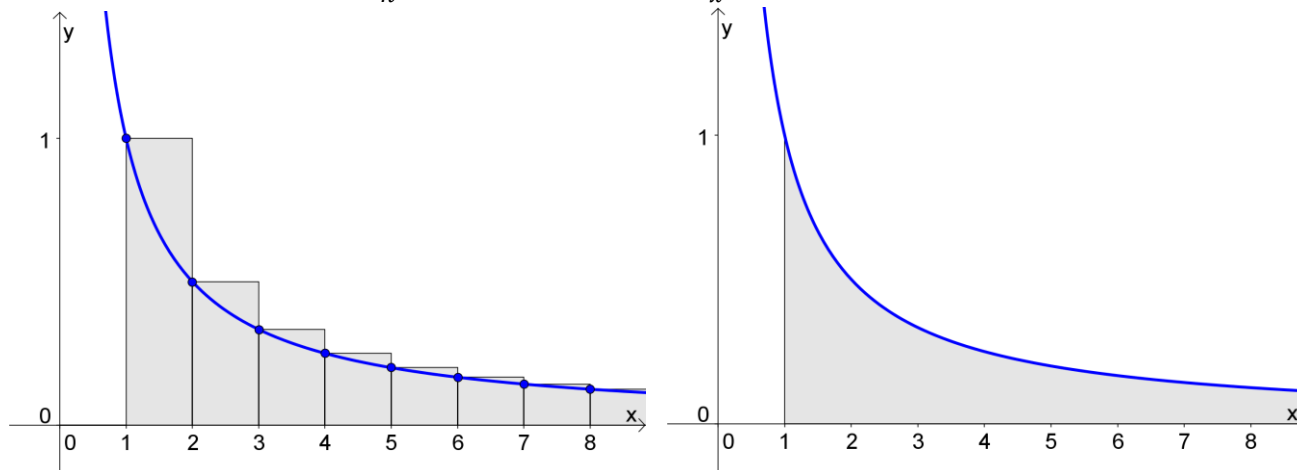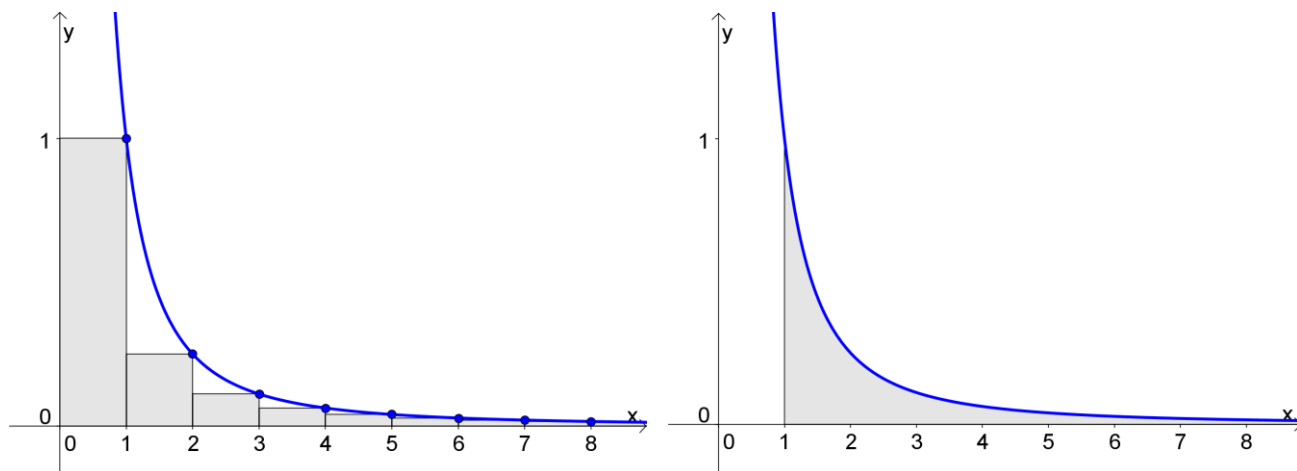# The Integral Test and Estimates of Sums

Let's look at the series $\sum_{n=1}^{\infty} \frac{1}{n}$ and compare it with $\int_1^{\infty} \frac{1}{x} dx$:

Note that $\int_1^{\infty} \frac{1}{x} dx < \sum_{n=1}^{\infty} \frac{1}{n}$.

So, by comparison, since $\int_1^{\infty} \frac{1}{x} dx$ _____, then the larger $\sum_{n=1}^{\infty} \frac{1}{n}$ must also _____.

Now let's look at the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and compare it with $\int_1^{\infty} \frac{1}{x^2} dx$:

Note that $\sum_{n=1}^{\infty} \frac{1}{n^2} < 1 + \int_1^{\infty} \frac{1}{x^2} dx$.

So, by comparison, since $\int_1^{\infty} \frac{1}{x^2} dx$ _____, then the smaller $\sum_{n=1}^{\infty} \frac{1}{n^2}$ must also _____.

The Integral Test (connects series with integrals)

Suppose that $a_n = f(n)$, where $f(x)$ is **continuous**, **positive**, and **decreasing** for all $x \geq N$ (N is some positive integer).

Then $\sum_{n=N}^{\infty} a_n$ and $\int_N^{\infty} f(x) dx$ both converge or both diverge.

Ex 1.

Does $\sum_{n=2}^{\infty} \frac{\ln n}{n}$ converge or diverge?

Ex 2.

Does $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converge or diverge?

Recall: $\int_1^\infty \frac{1}{x^p}\,dx$ converges if $p > 1$, and diverges if $p \le 1$.

Similarly, $\sum_{n=1}^\infty \frac{1}{n^p}$ (called a **p-series**) converges if $p > 1$, and diverges if $p \le 1$. (Proof on last page.)
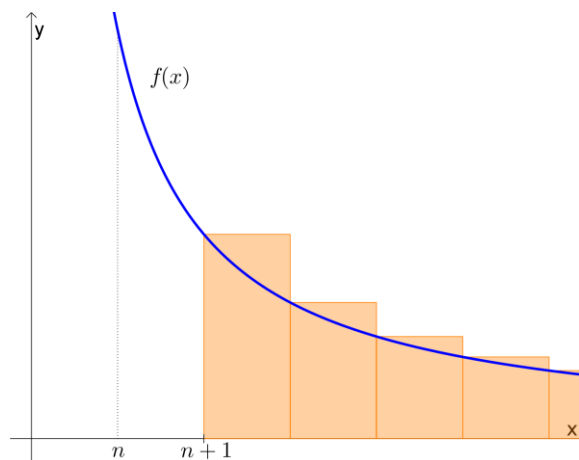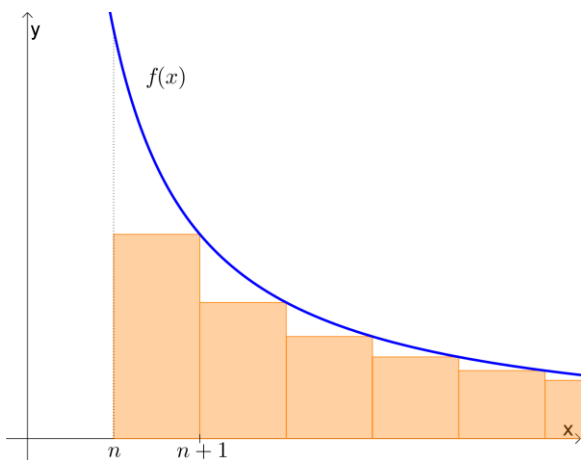
ex: $\sum_{n=1}^\infty \frac{1}{n^2}$ converges.   ex: $\sum_{n=1}^\infty \frac{1}{\sqrt{n}}$ diverges.

Estimates of Sums

How accurate is a given partial sum? In other words, how close is the sum of the first n terms (s_n) to the sum of the infinite number of terms ($\sum a_n$)?

$$\sum a_n = a_1 + a_2 + \cdots + a_n + a_{n+1} + a_{n+2} + \cdots$$

Remainder Estimate for the Integral Test

Suppose that $a_n = f(n)$, where $f(x)$ is continuous, positive, and decreasing for all $x \ge n$.

If $\sum a_n$ converges, then

$$\int_{n+1}^\infty f(x)\,dx \le R_n \le \int_n^\infty f(x)\,dx$$

Ex 3.

Estimate the error in using s_{10} as an approximation to the sum of $\sum_{n=1}^\infty \frac{1}{n^3}$. Also, how many terms are needed to make sure that the sum is accurate to within 0.0005?

If we take

$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_{n}^{\infty} f(x)\,dx$$

and add s_n to all three parts, we get

$$s_n + \int_{n+1}^{\infty} f(x)\,dx \le s_n + R_n \le s_n + \int_{n}^{\infty} f(x)\,dx.$$

Since $s_n + R_n = \sum a_n$, we get an upper and lower bound for our series:

$$\boldsymbol{s_n + \int_{n+1}^{\infty} f(x)\,dx \le \sum a_n \le s_n + \int_{n}^{\infty} f(x)\,dx}$$

(Note: This is referred to as $\boxed{3}$ in the book.)

Ex 4.

Use $\boxed{3}$ with $n = 10$ to give an improved estimate of the sum of $\sum_{n=1}^{\infty} \frac{1}{n^3}$ (better than s_{10}).

Proof of why $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$, and diverges if $p \le 1$.

Case 1 ($p = 0$): $\lim\limits_{n\to\infty} \frac{1}{n^p} = \lim\limits_{n\to\infty} \frac{1}{n^0} = \lim\limits_{n\to\infty} 1 = 1$, so $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges (by Test for Divergence)

Case 2 ($p < 0$): $\lim\limits_{n\to\infty} \frac{1}{n^p} = \lim\limits_{n\to\infty} n^{-p} = \infty$ since $-p > 0$. So, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges (by Test for Divergence)

Case 3 ($0 < p \le 1$): Note that $\frac{1}{x^p}$ is continuous, positive, and decreasing.

Since $\int_{1}^{\infty} \frac{1}{x^p}\,dx$ diverges for $0 < p \le 1$, we know $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges (by Integral Test)

Case 4 ($p > 1$): Note that $\frac{1}{x^p}$ is continuous, positive, and decreasing.

Since $\int_{1}^{\infty} \frac{1}{x^p}\,dx$ converges for $p > 1$, we know $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges (by Integral Test)

Thus, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$, and diverges if $p \le 1$. ∎

Practice

1. Does $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ converge or diverge? (Use the Integral Test. Be sure to show why the terms are decreasing.)

Challenge: Is there a "smallest" divergent series? (That is, a divergent series $\sum a_n$ where the terms a_n are smaller than the terms of any other divergent series.)

Q: A man leaves home and, after making three left turns, he ends up back at home, and finds two masked men waiting for him. What is happening?