

1. Write out the first few terms of the following series to show how the series starts. Then find the sum of the series.

$$\sum_{n=0}^{\infty} (-1)^n \frac{5}{4^n} = 5 - \frac{5}{4} + \frac{5}{16} - \dots$$

$$\begin{aligned} \uparrow & \quad \uparrow \\ a=5 & \quad ar = -\frac{5}{4} \\ & \quad 5r = -\frac{5}{4} \\ & \quad r = -\frac{1}{4} \end{aligned}$$

$$\begin{aligned} \text{Sum} &= \frac{5}{1 - (-\frac{1}{4})} \\ &= \frac{5}{\frac{5}{4}} \\ &= \boxed{4} \end{aligned}$$

$$\frac{a}{1-r}$$

2. Rewrite $3.1\overline{23} = 3.12323232323 \dots$ using a geometric series, and then express it as the ratio of two integers.

$$3.1\overline{23} = 3.1 + 0.023 + 0.00023 + 0.0000023 + \dots$$

$$= \frac{31}{10} + \frac{23}{1000} + \frac{23}{100000} + \frac{23}{10000000} + \dots$$

$$a = \frac{23}{1000}, \quad r = \frac{1}{100}$$

$$\text{Converges to } \frac{a}{1-r} = \frac{\frac{23}{1000}}{1 - \frac{1}{100}} = \frac{23}{990}$$

$$\begin{aligned} &= \frac{31}{10} + \frac{23}{990} \\ &= \boxed{\frac{1546}{495}} \end{aligned}$$

3. Does the following series converge or diverge? If it converges, find its sum.

$$\sum_{n=1}^{\infty} \frac{n}{2n+5}$$

$$\lim_{n \rightarrow \infty} \frac{n}{2n+5} = \lim_{n \rightarrow \infty} \frac{1}{2 + \frac{5}{n}} = \frac{1}{2} \neq 0$$

$$\text{So, } \sum_{n=1}^{\infty} \frac{n}{2n+5} \quad \boxed{\text{diverges}} \quad (\text{by Test for Divergence})$$

4. Does the following series converge or diverge? If it converges, find its sum. (Hint: Use log properties first to rewrite $\ln \frac{\sqrt{n}}{\sqrt{n+1}}$.)

$$\sum_{n=1}^{\infty} \ln \frac{\sqrt{n}}{\sqrt{n+1}} = \sum_{n=1}^{\infty} (\ln \sqrt{n} - \ln \sqrt{n+1}) = \sum_{n=1}^{\infty} \left(\frac{1}{2} \ln n - \frac{1}{2} \ln(n+1) \right)$$

$$S_n = \left(\frac{1}{2} \ln 1 - \frac{1}{2} \ln 2 \right) + \left(\frac{1}{2} \ln 2 - \frac{1}{2} \ln 3 \right) + \left(\frac{1}{2} \ln 3 - \frac{1}{2} \ln 4 \right) + \dots + \left(\frac{1}{2} \ln n - \frac{1}{2} \ln(n+1) \right)$$

$$= -\frac{1}{2} \ln(n+1)$$

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} -\frac{1}{2} \ln(n+1) = -\infty$$

Thus, $\sum_{n=1}^{\infty} \ln \frac{\sqrt{n}}{\sqrt{n+1}}$ diverges.

(Note that the series diverged even though $\ln \frac{\sqrt{n}}{\sqrt{n+1}} \rightarrow 0$ as $n \rightarrow \infty$.)

Challenge: Find the value for b for which $1 + e^b + e^{2b} + e^{3b} + \dots = 9$.

Challenge: If $\sum a_n$ and $\sum b_n$ both diverge, must $\sum(a_n + b_n)$ diverge?

Q: What has many keys but can't open any doors?

Challenge:

$$1 + e^b + e^{2b} + e^{3b} + \dots = \underbrace{1 + e^b + (e^b)^2 + (e^b)^3 + \dots}_{\substack{\text{geometric series} \\ a=1, r=e^b}}$$

We want

$$\frac{a}{1-r} = 9$$

$$\frac{1}{1-e^b} = 9 \quad \leftarrow \text{solve for } b$$

$$\frac{1}{9} = 1 - e^b$$

$$e^b = \frac{8}{9}$$

$$\boxed{b = \ln \frac{8}{9}}$$

Challenge:

$\boxed{\text{No}}$

For example, $a_n = (-1)^n$ $\{-1, 1, -1, 1, \dots\}$

$b_n = (-1)^{n+1}$ $\{1, -1, 1, -1, \dots\}$

$a_n + b_n$ $\{0, 0, 0, 0, \dots\}$

Here, both $\sum a_n$ and $\sum b_n$ diverge, but $\sum(a_n + b_n)$ converges (to 0).