

Sequences

An **infinite sequence** is an ordered list of numbers $a_1, a_2, a_3, \dots, a_n, \dots$

ex: 3, 7, 11, 15, 19, ...

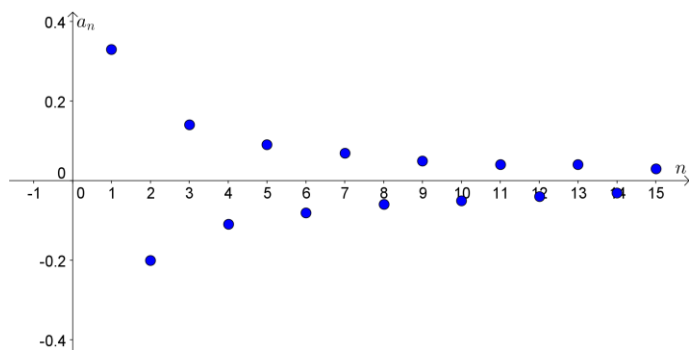
Other notations for a sequence a_1, a_2, a_3, \dots : $\{a_1, a_2, a_3, \dots\}$ $\{a_n\}$ $\{a_n\}_{n=1}^{\infty}$

ex: Suppose $a_n = \frac{n}{n+1}$ gives a formula for the sequence $\{a_n\}$. Then $\{a_n\} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\right\}$.

Ex 1.

Find a_1, a_2, a_3 , and a_4 if $a_n = \frac{(-1)^{n+1}}{2n+1}$.

We can visualize sequences as a bunch of dots on a coordinate system, with n on the x -axis, and a_n on the y -axis.



In the above graph, it looks like $\lim_{n \rightarrow \infty} a_n = \underline{\hspace{2cm}}$. Since the limit exists, we say that a_n converges. If the limit didn't exist, we'd say a_n diverges (ex: $\{n^2\}$ diverges).

Definition: (formal)

$\{a_n\}$ **converges to L** if for every $\epsilon > 0$ there is an integer N such that for all $n > N$, $|a_n - L| < \epsilon$.

Ex 2.

Does $\left\{\frac{1}{n}\right\}$ converge or diverge?

Ex 3.

Does $\{1, -1, 1, -1, 1, -1, 1, -1, \dots\}$ converge or diverge? Find a formula for the sequence.

Ex 4.

Find a formula for the n th terms of the sequence $1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, \dots$

All the same limit rules that we know and love still apply. So, if $\lim_{n \rightarrow \infty} a_n = A$ and $\lim_{n \rightarrow \infty} b_n = B$,

$$\lim_{n \rightarrow \infty} (a_n \pm b_n) = A \pm B$$

$$\lim_{n \rightarrow \infty} (k \cdot b_n) = k \cdot B$$

$$\lim_{n \rightarrow \infty} (a_n \cdot b_n) = A \cdot B$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{A}{B} \quad (\text{if } B \neq 0)$$

Ex 5.

Find the following limits.

$$\lim_{n \rightarrow \infty} \left(-\frac{1}{n}\right) =$$

$$\lim_{n \rightarrow \infty} \left(\frac{n-1}{n}\right) =$$

$$\lim_{n \rightarrow \infty} \left(\frac{4-7n^6}{n^6+3}\right) =$$

Ex 6.

Suppose $\{a_n\} = \{1, 2, 3, \dots\}$ and $\{b_n\} = \{-1, -2, -3, \dots\}$. Does $\{a_n + b_n\}$ converge or diverge?

The Squeeze Theorem for Sequences

If $a_n \leq b_n \leq c_n$ for all $n > N$, and if $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$, then $\lim_{n \rightarrow \infty} b_n = L$ also.

Ex 7.

Do the following sequences converge or diverge? If they converge, what do they converge to?

$$\left\{ \frac{\cos n}{n} \right\}$$

$$\left\{ \frac{1}{2^n} + 3 \right\}$$

$$\left\{ (-1)^n \frac{1}{n} \right\}$$

Theorem: (connects sequences and functions)

Suppose we have a sequence $\{a_n\}$ and a function $f(x)$ such that $a_n = f(n)$ for $n \geq n_0$. Then,

$$\lim_{x \rightarrow \infty} f(x) = L \quad \Rightarrow \quad \lim_{n \rightarrow \infty} a_n = L$$

Ex 8.

Does $\left\{ \frac{\ln n}{n} \right\}$ converge or diverge? If it converges, what does it converge to?

Note: Often, we'll just treat n as a continuous real variable (rather than a natural number), and just differentiate directly with respect to n .

Ex 9.

Does the sequence $a_n = \left(\frac{n+1}{n-1}\right)^n$ converge or diverge? If it converges, what does it converge to?

Recursive Definitions

A recursively defined sequence is one where each term depends on one or more terms before it.

Ex 10.

Write the first four terms of the sequence defined below.

$$a_1 = 1, \quad a_n = a_{n-1} + 2$$

Ex 11.

Assume that the following sequence converges and find its limit.

$$a_1 = -1, \quad a_{n+1} = \frac{a_n + 6}{a_n + 2}$$

$\{a_n\}$ is **increasing** if $a_n < a_{n+1}$ for all n .

$\{a_n\}$ is **decreasing** if $a_n > a_{n+1}$ for all n .

$\{a_n\}$ is **monotonic** if it is either increasing or decreasing.

$\{a_n\}$ is **bounded above** if there exists an M such that $a_n \leq M$ for all n .

$\{a_n\}$ is **bounded below** if there exists an m such that $a_n \geq m$ for all n .

$\{a_n\}$ is **bounded** if it is bounded from above and below.

Monotonic Sequence Theorem

If $\{a_n\}$ is bounded and monotonic, then $\{a_n\}$ converges.

Practice

1. Write the first four terms of the sequence defined below. (This generates the **Fibonacci numbers**.)

$$a_1 = 1, \quad a_2 = 1, \quad a_{n+1} = a_n + a_{n-1}$$

2. Find a formula for the n th terms of the sequence $5, \frac{3}{2}, \frac{1}{3}, -\frac{1}{4}, -\frac{3}{5}, -\frac{5}{6}, -1, \dots$

3. Which of the following sequences converge, and which diverge? Find the limit of each convergent sequence.

a) $a_n = (-1)^n \left(1 - \frac{1}{n}\right)$

b) $a_n = 2 + (-1)^n$

c) $a_n = \frac{2n+1}{1-3\sqrt{n}}$

$$d) a_n = \left(\frac{2}{n}\right)^{1/n}$$

$$e) a_n = \frac{1 - \cos n}{n^2}$$

Q: What are the next two letters in this sequence: A E F H I K L M ?