Taylor and Maclaurin Series

If a function has a power series representation, what does it look like?

$f(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots$ Find c_0 by plugging in 0: $f(0) = c_0$

$f'(x) = c_1 + 2c_2x + 3c_3x^2 + 4c^4x^3 + \cdots$ Find c_1 by plugging in 0: $f'(0) = c_1$

$f''(x) = 2c_2 + 3 \cdot 2c_3x + 4 \cdot 3c_4x^2 + \cdots$ Find c_2 by plugging in 0: $f''(0) = 2c_2 \rightarrow c_2 = \frac{f''(0)}{2}$

$f'''(x) = 3 \cdot 2c_3 + 4 \cdot 3 \cdot 2c_4x + \cdots$ Find c_3 by plugging in 0: $f'''(0) = 3 \cdot 2c_3 \rightarrow c_3 = \frac{f'''(0)}{3 \cdot 2}$

Find c_4: $f^{(4)}(0) = 4 \cdot 3 \cdot 2c_4 \quad \rightarrow \quad c_4 = \frac{f^{(4)}(0)}{4 \cdot 3 \cdot 2}$

In general, $c_n = \frac{f^{(n)}(0)}{n!}$

So, $\boldsymbol{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n} = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$

This is called the **Taylor series of f at 0**. (Also called the **Maclaurin series of f**.)

We could also put the center at $x = a$:

$\boldsymbol{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n} = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$

This is called the **Taylor series of f at a**.

Ex 1.

Find the Maclaurin series of $f(x) = e^x$ using the definition of a Maclaurin series. Also find the radius of convergence.

Ex 2.

Find the Taylor series for $f(x) = e^x$ at $a = 2$, given your results from the previous example.

Ex 3.

Find the Maclaurin series for $\sin x$ using the definition of a Maclaurin series.

Ex 4.

Find the Maclaurin series for $\cos x$, given your results from the previous example.

Ex 5.

Find the Maclaurin series for $x \cos x$, given your results from the previous example.

Here is the Taylor series generated by $f(x) = (1+x)^k$ with convergence on $-1 < x < 1$:

$$\boldsymbol{(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots} \quad \text{(the Binomial Series)}$$

The above is true where k is *any real number*. Also, $\binom{k}{n} = \frac{k(k-1)(k-2)\ldots(k-n+1)}{n!}$

Ex 6.

Find the Maclaurin series for $\frac{1}{\sqrt{4-x}}$.

Here are some frequently-used Taylor Series.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \qquad |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \qquad -1 < x \le 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \qquad |x| \le 1$$

Taylor series can help us find integrals and evaluate limits.

Ex 7.

Find an infinite series representation for $\int e^{-x^2}\,dx$.

Ex 8.

Evaluate $\int_0^1 e^{-x^2}\,dx$ correct to within an error of 0.001.

Ex 9.

Evaluate $\lim_{x\to 0}\frac{\tan^{-1}x-x}{x^3}$.

Note: We can multiply and divide power series as follows.

Ex 10.
Find the first three nonzero terms in the Maclaurin series for $e^x \sin x$.

Ex 11.
Find the first three nonzero terms in the Maclaurin series for $\tan x$.

Practice

1. Find the Taylor series for $\frac{1}{x}$ at $x = 2$ using the definition of a Taylor series.

2. Find a Maclaurin series for $f(x) = x^2 \ln(1 + x^3)$ by using $\ln(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n}$.

Q: When can you add two to eleven and get one as the correct answer?