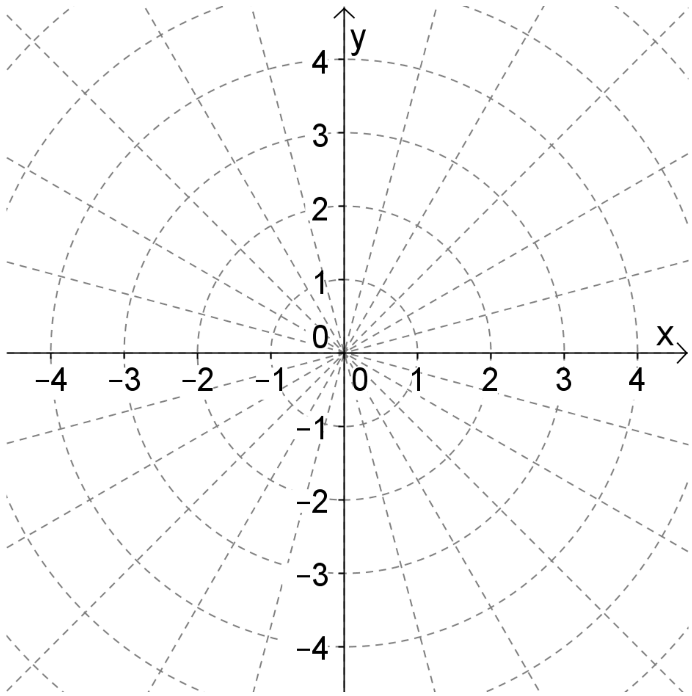
Polar Coordinates

Recall from trigonometry that a polar coordinate system is another way to locate points on plane. So, instead of Cartesian coordinates (x, y), we have polar coordinates (r, θ).



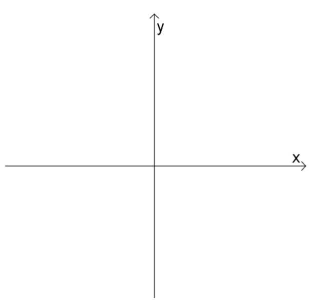
Notes:

Each (x, y) point has an infinite number of polar coordinate representations (ex: $(1,0) = (1,2\pi k)$).
r can be negative (ex: $(1,0) = (-1, \pi)$).

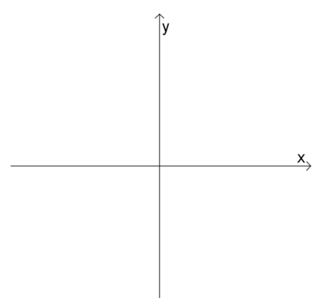
Ex 1.

Graph the following polar equations.

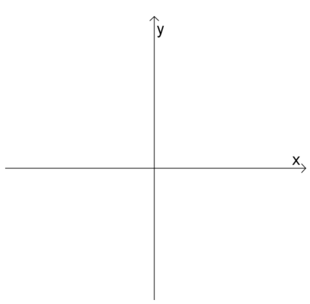
$r = 1$



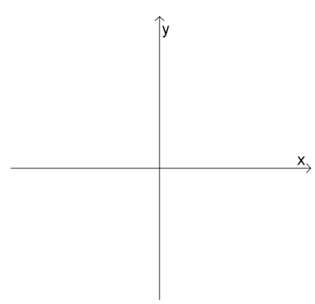
$r = -2$



$\theta = \frac{\pi}{6}$



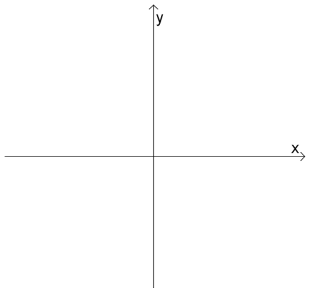
$\theta = \frac{7\pi}{6}$



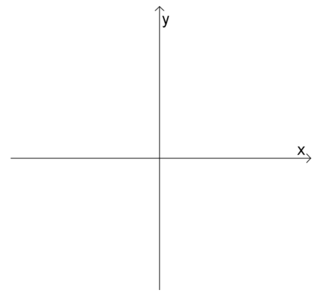
Ex 2.

Graph the following regions.

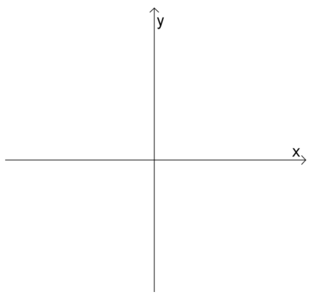
$1 \le r \le 3$



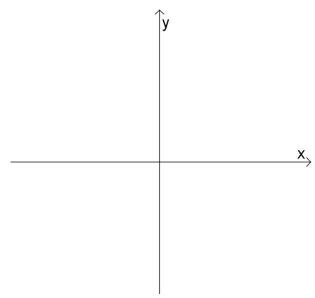
$1 \le r \le 3$ and $0 \le \theta \le \frac{\pi}{2}$



$-3 \le r < 2$ and $\theta = \frac{\pi}{4}$



$\frac{2\pi}{3} \le \theta \le \frac{5\pi}{6}$



Here are the equations to get you from and to polar coordinates:

$x = r\cos\theta$, $y = r\sin\theta$ $r^2 = x^2 + y^2$, $\tan\theta = \frac{y}{x}$

Ex 3.

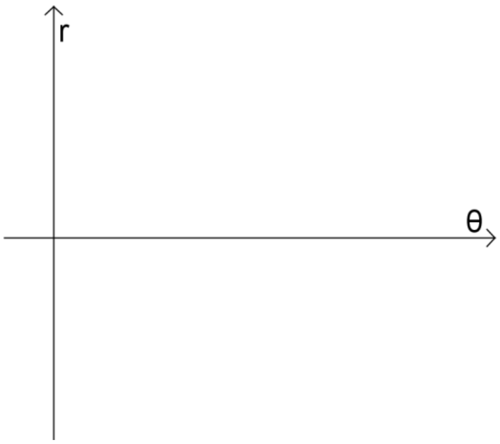
Find a polar equation for $x^2 + (y-3)^2 = 9$.

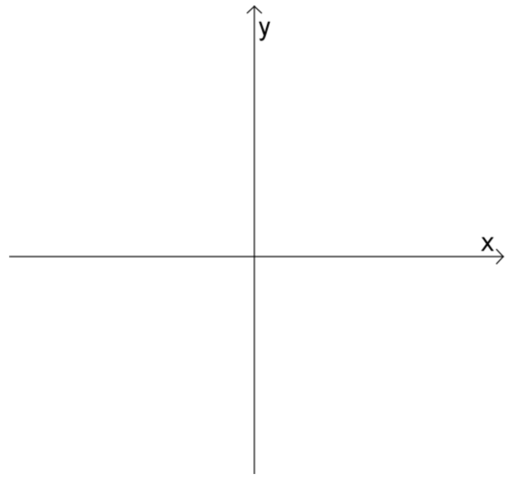
Ex 4.

Replace $r = \frac{4}{2\cos\theta - \sin\theta}$ with an equivalent Cartesian equation.

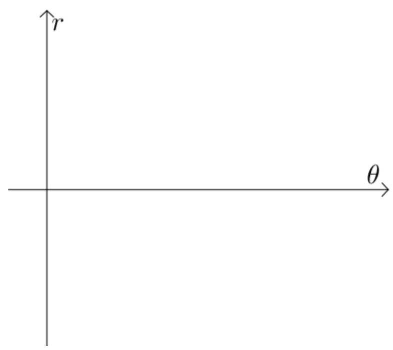
Making a sketch in the Cartesian $r\theta$-plane can help graph polar curves when you can't find an equivalent Cartesian equation. That's the main technique we'll use here.

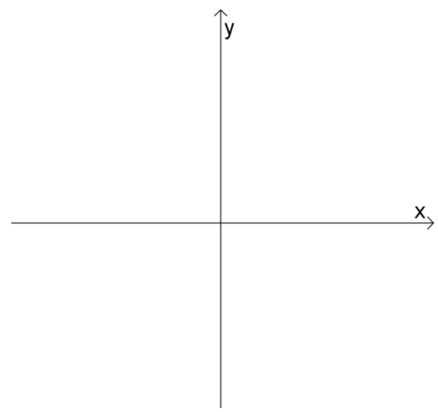
Ex 5.
Graph the curve $r = 1 - \cos\theta$.





Ex 6.
Graph the curve $r = 2 + 3\sin\theta$.





Symmetries

Noticing symmetries can also help when graphing in polar coordinates.

If you can get back the original equation after…
…replacing θ by $-\boldsymbol{\theta}$, then graph is symmetric about $\boldsymbol{x}$**-axis**.
…replacing θ by $\boldsymbol{\pi - \theta}$, then graph is symmetric about $\boldsymbol{y}$**-axis**.
…replacing r by $-\boldsymbol{r}$, then graph is symmetric about **origin**.

Some useful trig identities:
$\sin(-\theta) = -\sin\theta$
$\boldsymbol{\cos(-\theta) = \cos\theta}$
$\boldsymbol{\sin(\pi - \theta) = \sin\theta}$
$\cos(\pi - \theta) = -\cos\theta$

In Ex 5 ($r = 1 - \cos\theta$), we can see the symmetry about the x-axis by replacing θ with $-\theta$:
$r = 1 - \cos(-\theta)$
$r = 1 - \cos\theta$

Slope

To find the slope of a polar equation $r = f(\theta)$, we can use the parametric equations:

$x = r\cos\theta$ and $y = r\sin\theta$ (the parameter here is θ)

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

For example, what is the slope of $r = 2 + 3\sin\theta$ (from Ex 6) at $\theta = \frac{\pi}{2}$?

$x = r\cos\theta = (2 + 3\sin\theta)\cos\theta$

Using Product Rule, $\frac{dx}{d\theta} = (2 + 3\sin\theta)(-\sin\theta) + 3\cos\theta\,(\cos\theta) = -2\sin\theta - 3\sin^2\theta + 3\cos^2\theta$

$y = r\sin\theta = (2 + 3\sin\theta)\sin\theta$

Using Product Rule, $\frac{dy}{d\theta} = (2 + 3\sin\theta)(\cos\theta) + 3\cos\theta\,(\sin\theta) = 2\cos\theta + 6\sin\theta\cos\theta$

So, $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2\cos\theta + 6\sin\theta\cos\theta}{-2\sin\theta - 3\sin^2\theta + 3\cos^2\theta}$.

At $\theta = \frac{\pi}{2}$, $\frac{dy}{dx} = \frac{2\cos\frac{\pi}{2} + 6\sin\frac{\pi}{2}\cos\frac{\pi}{2}}{-2\sin\frac{\pi}{2} - 3\sin^2\frac{\pi}{2} + 3\cos^2\frac{\pi}{2}} = \frac{0}{-5} = \boxed{0}$

Notes:

Horizontal tangent lines will happen when $\frac{dy}{d\theta} = 0$ and $\frac{dx}{d\theta} \neq 0$.

Vertical tangent lines will happen when $\frac{dx}{d\theta} = 0$ and $\frac{dy}{d\theta} \neq 0$.

If both $\frac{dx}{d\theta} = 0$ and $\frac{dy}{d\theta} = 0$ at $\theta = \theta_0$, then you'll have to check $\lim_{\theta \to \theta_0} \frac{dy}{dx}$ and possibly use L'Hospital.

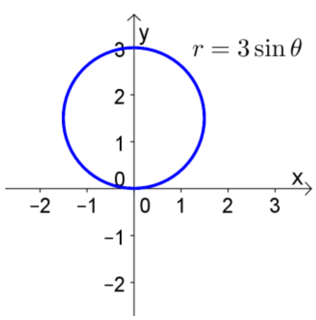
Ex 7.

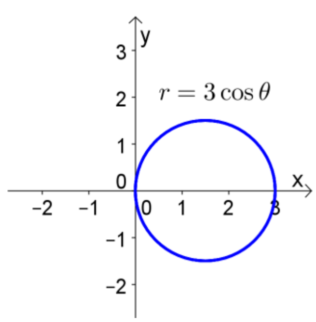
Find the values of θ in $[0,2\pi)$ where the tangent line of $r = 1 - \cos\ \theta$ (Ex 5) is horizontal or vertical.

Common Polar Equations and Graphs

Circle

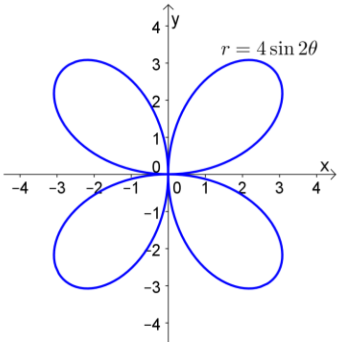
$r = a\sin\theta$ and $r = a\cos\theta$

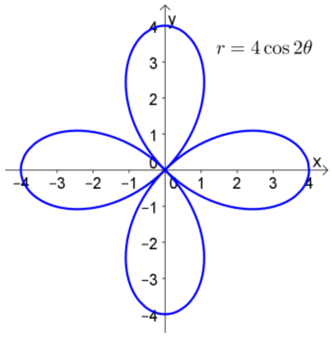


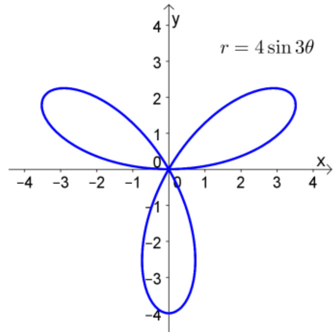


Rose

$r = a \sin n\theta$ and $r = a \cos n\theta$



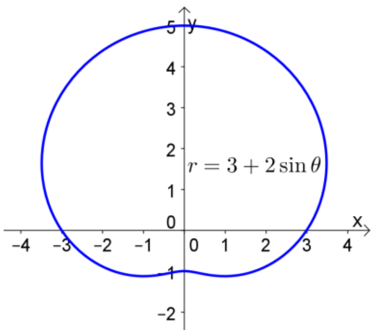


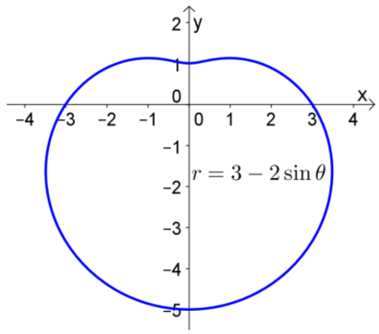


Limacon (pronounced "LEE-ma-sahn")

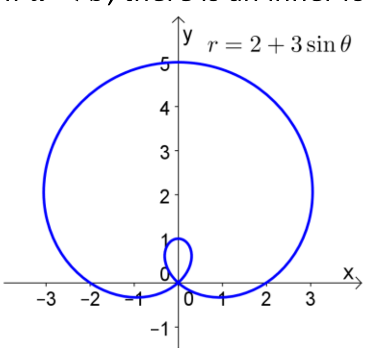
$r = a \pm b \sin\theta$ and $r = a \pm b \cos\theta$

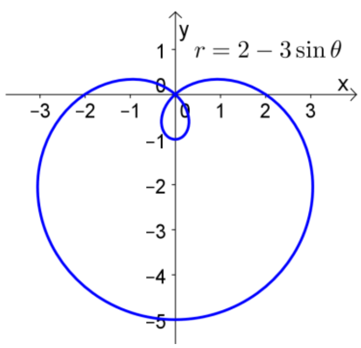
If $a > b$, there is no inner loop.



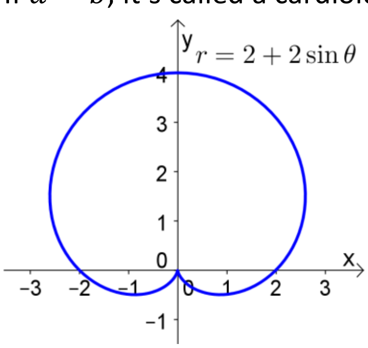


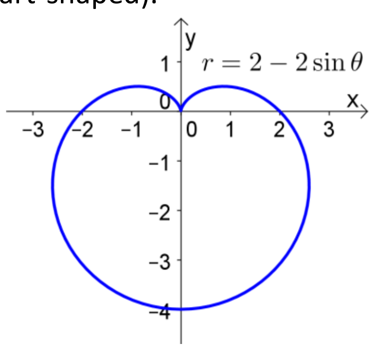
If $a < b$, there is an inner loop.





If $a = b$, it's called a cardioid (heart-shaped).

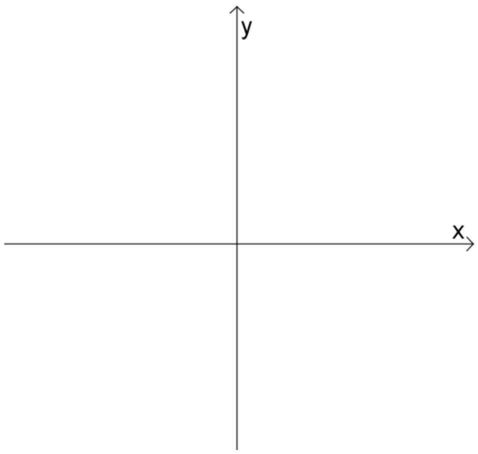




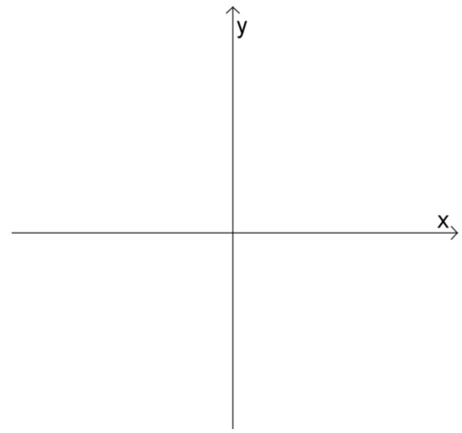
Practice

1. Graph the following regions.

$\frac{4\pi}{3} \leq \theta \leq \frac{5\pi}{3}$ and $r \geq 1$

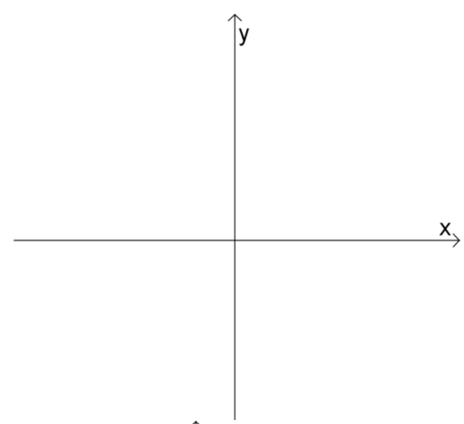


$0 \leq \theta \leq \frac{3\pi}{2}$ and $r = -4$

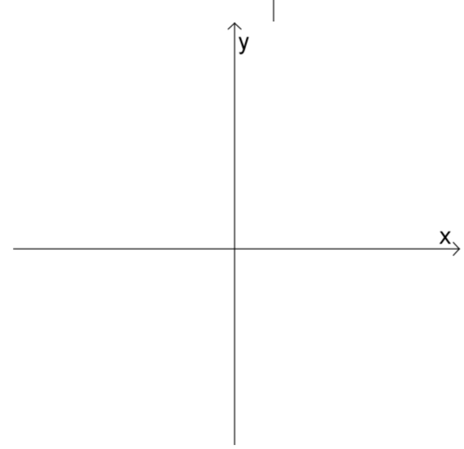


2. Find a polar equation (that has been solved for r) for $x^2 + y^2 = 9$.

3. Replace $r^2 = 4r\cos\theta$ with an equivalent Cartesian equation.
   Graph it.



4. Replace $r = -3\csc\theta$ with an equivalent Cartesian equation.
   Graph it.



5. Graph the curve $r = 1 + \sin\theta$.

6. Find the slope of the curve $r = \cos 2\theta$ at $\theta = 0, \frac{\pi}{2}, -\frac{\pi}{2}$, and π. Then graph the curve.

Q: Suppose your boyfriend/girlfriend sends you this text message:
A,B,C,D,E,F,G,H,I,J,K,L,M,N,O,P,Q,R,S,T,V,W,X,Y,Z.
What does the message mean?