Calculus with Parametric Curves

If we have a curve in parametric form, we can still calculate the slope of the tangent curve.

Starting with the Chain Rule, we have $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$. Solving for $\frac{dy}{dx}$, we get:

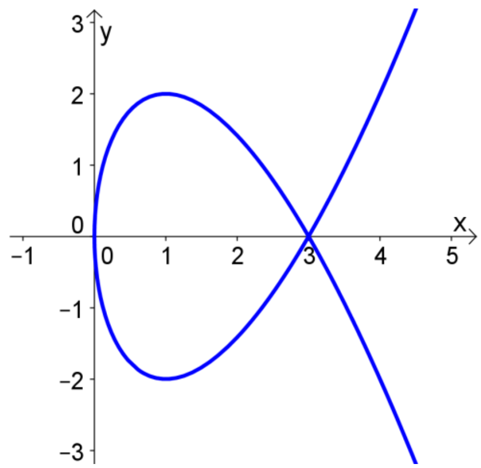
$$\boldsymbol{\frac{dy}{dx} = \frac{dy/dt}{dx/dt}}$$

For the second derivative, we compute: $\frac{d^2y}{dx^2} = \frac{d}{dx}(y') = \frac{dy'/dt}{dx/dt}$. So,

$$\boldsymbol{\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} \qquad \left(\text{where } y' = \frac{dy}{dx}\right)}$$

Ex 1.

Find the tangent to the curve $x = t^2,\ y = t^3 - 3t$ when $t = \sqrt{3}$.



(Note that there are two tangent lines at the point $(3,0)$: one at $t = \sqrt{3}$ and one at $t = -\sqrt{3}$.)

Ex 2.

Find $\frac{d^2y}{dx^2}$ for $x = t^2,\ y = t^3 - 3t$.

To find the **net area** under the curve with parametric equations $x = f(t),\ y = g(t)$, we compute:

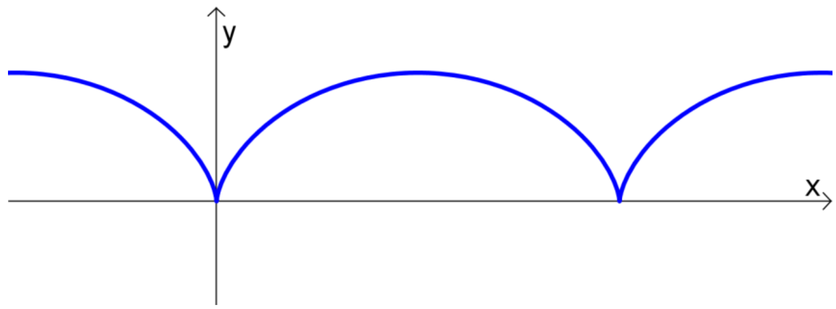
Net area $= \int y\,dx = \int_{t=a}^{t=b} \boldsymbol{g(t)f'(t)dt}$

Notes: When a curve is *below* the x-axis, areas are signed *negative*, as expected.
As you go from $t = a$ to $t = b$, the curve must be traversed exactly once.

Ex 3.

Find the area under one arch of the following curve (called a cycloid):

$x = 2(t - \sin t),\ y = 2(1 - \cos t)$



Note:

If x decreases as t increases, then this will sign the area negative.

So, if you want positive area, always make sure x is increasing.

ex: Consider $x = \cos t, y = 1$.

If we set up the integral like $\int_{t=0}^{t=\pi} 1 \cdot (-\sin t)\,dt$, then you'll get a negative area:

$\int_{t=0}^{t=\pi} 1 \cdot (-\sin t)\,dt = [\cos t]_0^{\pi} = -1 - 1 = -2.$

This is because as t goes from 0 to π, x goes from 1 to -1 (i.e. it decreases).

Instead, we'd want to set up the integral like this: $\int_{t=\pi}^{t=0} 1 \cdot (-\sin t)\,dt = \cdots = 2$

The **arc length** for $x = f(t),\ y = g(t)$ from $t = a$ to $t = b$ is:

$$\textbf{Arc length} = \int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Note: As you go from $t = a$ to $t = b$, the curve must be traversed exactly once.

The **surface area** for the curve $x = f(t),\ y = g(t)$ from $t = a$ to $t = b$ revolved about the x-axis is:

$$\textbf{Surface area} = \int_{t=a}^{t=b} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

When revolving about the y-axis, it is:

$$\textbf{Surface area} = \int_{t=a}^{t=b} 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Note: As you go from $t = a$ to $t = b$, the curve must be traversed exactly once.

Practice

1. Find an equation for the line tangent to the curve $x = \sec t,\ y = \tan t, -\frac{\pi}{2} < t < \frac{\pi}{2}$ at the point where $t = \frac{\pi}{4}$.

2. Find the length of the curve $x = r\cos t\,,\ y = r\sin t\,,\ 0 \le t \le 2\pi$ (using the arc length formula). What have you just done?

3. Find the area of the surface generated by revolving $x = \cos t\,,\ y = 1 + \sin t\,,\ 0 \le t \le 2\pi$ about the x-axis.

Q: April says May is a liar. May says June is a liar. June says April and May are both liars. If only one person is telling the truth, who is it?