Math 181 - Test #3 Info and Review Exercises

Fall 2017, Prof. Beydler

Test Info

- Date: Wednesday, November 29, 2017
- Will cover sections 9.1, 9.3, 10.1-10.4, 11.1-11.5.
- You'll have the entire class to finish the test.
- For this test, you'll need a **scientific calculator**.
- No notes, no books, no phones, no smart watches during the test.
- There will be a seating chart for the test.
- Where to get help as you're studying:
    - Office hours
    - TMARC, LAC, or other tutoring centers
    - E-mail me at dbeydler@mtsac.edu

Here are **some** of the formulas/concepts that you'll need to know:

Parametric Curves

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$

$$\text{Net area} = \int y\,dx = \int_{t=a}^{t=b} g(t)f'(t)dt$$

$$\text{Arc length} = \int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$\text{Surface area} = \int_{t=a}^{t=b} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \quad \text{or} \quad \int_{t=a}^{t=b} 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Polar Coordinates

$x = r\cos\theta\,,\ \ y = r\sin\theta$ $\qquad\qquad$ $r^2 = x^2 + y^2,\ \ \tan\theta = \frac{y}{x}$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Horizontal tangent lines will happen when $\frac{dy}{d\theta} = 0$ and $\frac{dx}{d\theta} \neq 0$.

Vertical tangent lines will happen when $\frac{dx}{d\theta} = 0$ and $\frac{dy}{d\theta} \neq 0$.

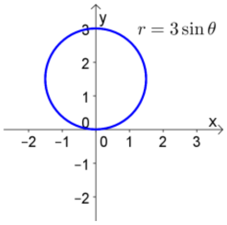
If both $\frac{dx}{d\theta} = 0$ and $\frac{dy}{d\theta} = 0$ at $\theta = \theta_0$, then you'll have to check $\lim\limits_{\theta\to\theta_0} \frac{dy}{dx}$ and possibly use L'Hospital.

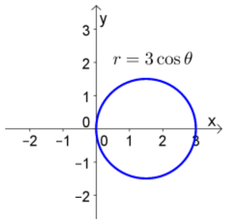
$$\text{Area} = \int_{\theta=\alpha}^{\theta=\beta} \frac{1}{2} r^2 d\theta$$

$$\text{Arc length} = \int_{\theta=\alpha}^{\theta=\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

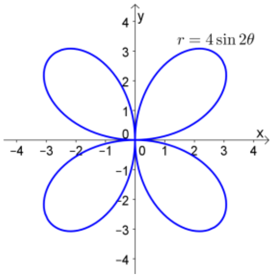
<u>Know these polar graphs well!!</u>

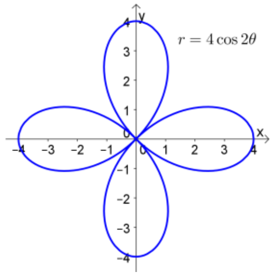
Circle: $r = a\sin\theta$ and $r = a\cos\theta$

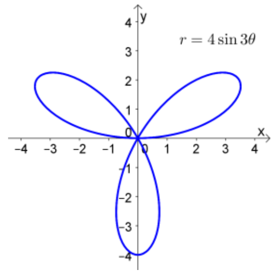




Rose: $r = a\sin n\theta$ and $r = a\cos n\theta$

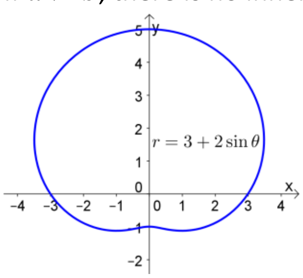


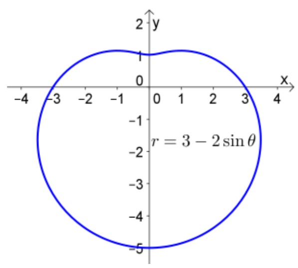




Limacon: $r = a \pm b\sin\theta$ and $r = a \pm b\cos\theta$

If $a > b$, there is no inner loop.





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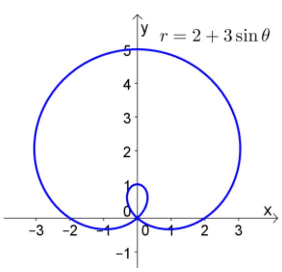
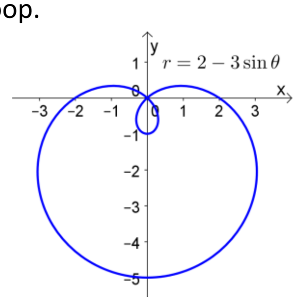
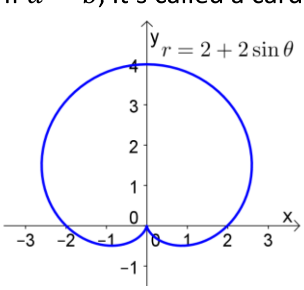
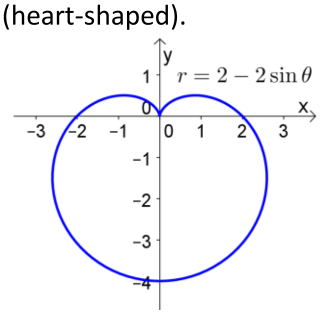




If $a = b$, it's called a cardioid (heart-shaped).





Series

Geometric series: $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$ converges to $\frac{a}{1-r}$ if $|r| < 1$, diverges if $|r| \geq 1$

p-series: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$, and diverges if $p \leq 1$

Does a series converge or diverges? Here are the tests we've learned so far…

Test for Divergence: If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

The Integral Test: Suppose that $a_n = f(n)$, where $f(x)$ is **continuous**, **positive**, and **decreasing** for all $x \geq N$.
Then $\sum_{n=N}^{\infty} a_n$ and $\int_N^{\infty} f(x)\,dx$ both converge or both diverge.

The Comparison Test: Suppose a_n and b_n have nonnegative terms, and N is some integer.
If $a_n \leq b_n$ for all $n > N$ and if $\sum b_n$ converges, then the smaller $\sum a_n$ also converges.
If $b_n \leq a_n$ for all $n > N$ and if $\sum b_n$ diverges, then the bigger $\sum a_n$ also diverges.

The Limit Comparison Test: Suppose a_n and b_n have positive terms for all $n \geq N$ (N is some integer).

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Alternating Series Test (AST): $\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots$ converges if:

1. $b_n > 0$
2. $b_{n+1} \leq b_n$ for all $n \geq N$
3. $b_n \to 0$

How well does a partial sum approximate the infinite sum?

Remainder Estimate for the Integral Test: Suppose that $a_n = f(n)$, where $f(x)$ is continuous, positive, and decreasing for all $x \geq n$. If $\sum a_n$ converges, then $\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_n^{\infty} f(x)\,dx$

Alternating Series Estimation Theorem

Suppose we have an alternating series $s = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$ where $b_n > 0$, $b_{n+1} \leq b_n$, and $b_n \to 0$.
Then, $|R_n| = |s - s_n| \leq b_{n+1}$

I'll give you these formulas if you need them:

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

$$\int \csc x\,dx = -\ln|\csc x + \cot x| + C$$

3: $s_n + \int_{n+1}^{\infty} f(x)\,dx \leq \sum a_n \leq s_n + \int_n^{\infty} f(x)\,dx$

Review Exercises

Note: If you write up the answers to all of the review exercises listed below, and hand them in at the test, you can earn up to 3% extra credit towards your test (depending on neatness and completeness)! It is important to understand that these review exercises are not guaranteed to cover all of the potential problems on the test. Please review the notes, practice problems, previous quizzes, and homework problems to fully prepare for the test.

1. Show that $y = Ae^{x} + Be^{-5x}$ is a solution to the following differential equation. (Note: A and B are constants.)
$y'' + 4y' - 5y = 0$

2. Show that $y = -te^{-t} + 6$ is a solution to the following initial-value problem. (Note: Here, you have to show that the DE and **both** initial conditions are satisfied.)
$y'' + 2y' = te^{-t}, \quad y(0) = 6, \quad y'(0) = -1$

3. The following DE models the temperature $T(t)$ of an object: $\frac{dT}{dt} = -k(T - T_S)$. This is called Newton's Law of Cooling. T_S is the surrounding temperature, and k is a positive constant that depends on the type of object.
   a. For what values of T will the temperature be decreasing?
   b. For what values of T will the temperature be increasing?
   c. For what value(s) of T will the temperature be constant? (In these cases, they're called equilibrium solutions.)

4. Find the general solution to each of the following differential equations.
   a. $\frac{dy}{dx} = \frac{x\sqrt{1-y^2}}{x^2+1}$
   b. $\frac{dy}{dx} = e^{y-x}$
   c. $y' + xy = 2x$

5. Solve each initial value problem.
   a. $y' = y^2 + 2xy^2, \quad y(0) = 2$
   b. $ye^{2x}\frac{dy}{dx} = x^2, \quad y(0) = -\frac{1}{\sqrt{2}}$

6. Find the orthogonal trajectories of the family of curves $y^2 = kx^3$, where k is an arbitrary constant.

7. Find the orthogonal trajectories of the family of curves $x^2 - y^2 = k$, where k is an arbitrary constant.

8. A 200-gallon vat is full of a solution that's 4% alcohol. A solution with 10% alcohol starts pouring into the vat at 20 gal/min. Assume the solution in the vat is kept well-mixed and drains from the vat at 20 gal/min. Find the percentage of alcohol in the vat's solution after 10 minutes.

9. Given the following parametric equations/intervals of a particle in the xy-plane, find the related Cartesian equation and graph it. Then, indicate the portion of the graph traced by the particle and the direction of motion.
   a. $x = 2 + \cos t\,,\ y = 3 + 2\sin t\,,\ \ 0 \le t \le 2\pi$
   b. $x = \cos 2t\,,\ y = \sin t\,, -\frac{\pi}{2} \le t \le \frac{\pi}{2}$

10. Find an equation for the line tangent to the curve $x = \frac{1}{2}t^2 + 1,\ y = \frac{1}{3}t^3 - t$ at the point where $t = 2$.

11. Find $\frac{d^2y}{dx^2}$ for $x = t + \cos t\,,\ y = 1 + \sin t$.

12. Find the length of the curve $x = \sqrt{t} - 2,\ y = 2t^{3/4},\ 1 \le t \le 16$

13. Find the area of the surface generated by revolving $x = e^t \cos t\,,\ y = e^t \sin t\,,\ 0 \le t \le \frac{\pi}{2}$ about the x-axis.

14. Find the area of the region…
    a. …enclosed by $r = 3\sin 2\theta$.
    b. …inside $r = 4 + 4\cos\theta$ and outside $r = 6$.
    c. …inside $r = 1 + \sin\theta$ and outside $r = \sin\theta$.
    d. …inside both $r = 1 + \cos\theta$ and $r = 3\cos\theta$.
    e. …inside both $r = 1$ and $r = 2\sin\theta$
    f. …within the inner loop of $r = 2 + 4\sin\theta$.

15. Find the length of the curve $r = e^{2\theta}, 0 \le \theta \le 2$.

16. Find the length of the curve $r = \sqrt{1 + \sin 2\theta},\ 0 \le \theta \le \pi\sqrt{2}$.

17. Find the slope of the curve $r = \cos\frac{\theta}{3}$ at $\theta = \pi$.

18. Find the values of θ in $[0,2\pi)$ where the tangent line of $r = 1 - \sin\ \theta$ is horizontal or vertical.

19. Determine whether each sequence converges or diverges. If it converges, find the limit.
    a. $a_n = \tan^{-1}\left(\ln\frac{1}{n}\right)$
    b. $a_n = \frac{\tan^{-1} n}{n}$
    c. $a_n = \sqrt{\frac{2n^3}{n^3+1}}$
    d. $a_n = \frac{3n^4+2n}{\sqrt{5n^7-n^2+1}}$
    e. $a_n = n^{1/n}$
    f. $a_n = \ln n - \ln(n+1)$
    g. $a_n = \frac{\ln(n+2)}{\sqrt{n}}$
    h. $a_n = \frac{5^n}{n!}$
    i. $a_n = \sqrt{n}\sin\frac{1}{\sqrt{n}}$
    j. $a_n = \frac{\sin^2 n + n}{n^2}$
    k. $a_n = \frac{2^n}{3^{n+2}}$

20. Assume that the following sequence converges and find its limit.
    $a_1 = 0, \quad a_{n+1} = \sqrt{5 + 4a_n}$

21. Assume that the following sequence converges and find its limit.
    $2\,, 2 + \frac{1}{2}, 2 + \frac{1}{2+\frac{1}{2}}, 2 + \frac{1}{2+\frac{1}{2+\frac{1}{2}}}, \ldots$

22. Determine whether each series is convergent or divergent.
   a. $\sum_{n=0}^{\infty} \frac{2n^2-3n+1}{\sqrt{5n^6+4n+2}}$
   b. $\sum_{n=1}^{\infty} \ln\left(\frac{n}{2n+1}\right)$
   c. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$
   d. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$
   e. $\sum_{n=2}^{\infty} \frac{4^{n+1}}{5^{n+2}}$ (If it converges, what does it converge to?)
   f. $\sum_{n=1}^{\infty} \frac{(-2)^n}{3^{2n+1}}$ (If it converges, what does it converge to?)
   g. $\sum_{n=1}^{\infty} \frac{\pi^{n-1}}{3^{n+2}}$ (If it converges, what does it converge to?)
   h. $\sum_{n=1}^{\infty} \sqrt[n]{3}$
   i. $\sum_{n=2}^{\infty} \frac{6\sqrt{n}-12}{3n^2+11}$
   j. $\sum_{n=1}^{\infty} \frac{(-1)^n+4}{n^2+n+1}$
   k. $\sum_{n=1}^{\infty} (-1)^{n+1} n e^{-n}$
   l. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{\sqrt{n^3+2}}$

23. Find the sum of each series.
   a. $\sum_{n=1}^{\infty} (\tan^{-1} n - \tan^{-1}(n+1))$
   b. $\sum_{n=1}^{\infty} \frac{2}{n(n+2)}$

24. Express $7.23\overline{45}$ as a ratio of integers.

25. Use $\sum_{n=1}^{\infty} \frac{1}{(2n+1)^3}$ to answer the parts below.
   a. Estimate the error in using s_5 as an approximation to the series' true sum.
   b. How many terms are needed to make sure that the sum is accurate to within 0.000005?
   c. Use $\boxed{3}$ with $n = 5$ to give an improved estimate of the series' sum (better than s_5).

26. Use $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ to answer the parts below.
   a. Estimate the error in using s_6 as an approximation to the series' true sum.
   b. How many terms are needed to make sure that the sum is accurate to within 0.05?
   c. Use $\boxed{3}$ with $n = 6$ to give an improved estimate of the series' sum (better than s_6).

27. Find the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n n!}$ correct to 4 decimal places.

28. Find the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5}$ correct to 4 decimal places.