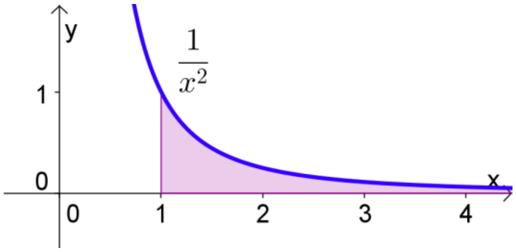
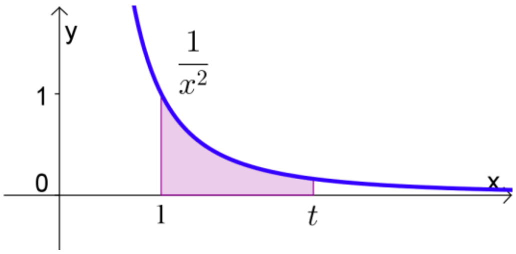
Improper Integrals





In general, these types of **improper integrals (Type I)** are defined like this:

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$

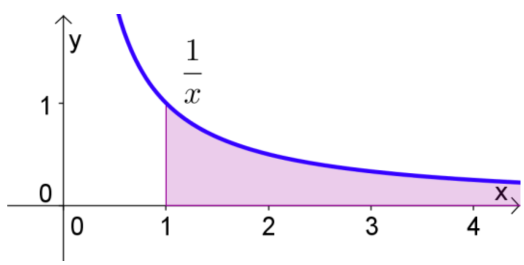
$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$$

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$$

In each case above, if the limit is finite, we say the improper integral _____________.
If the limit does not exist, the improper integral ________________.

Ex 1.

Does $\int_1^\infty \frac{1}{x}\,dx$ converge or diverge?



Ex 2.

Evaluate:

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

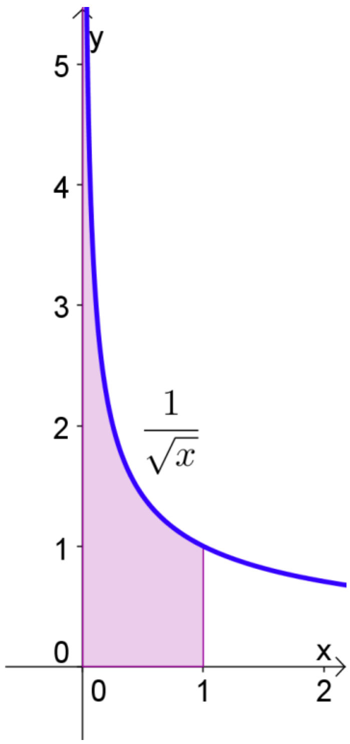
Ex 3.

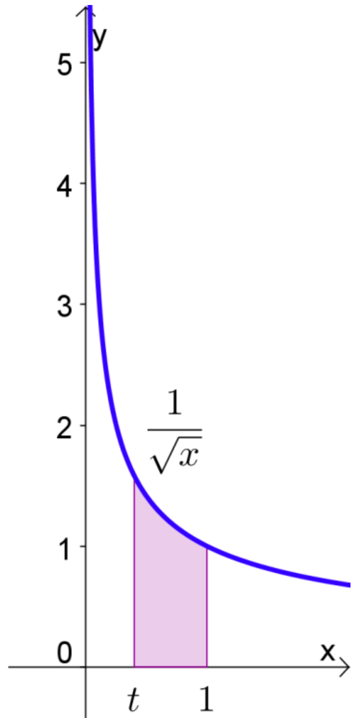
For what values of p does the integral $\int_1^\infty dx/x^p$ converge? When the integral does converge, what is its value?

$\int_1^\infty \frac{1}{x^p}\,dx$…

…converges to $\frac{1}{p-1}$ if $p > 1$.

…diverges if $p \leq 1$.





In general, these types of **improper integrals (Type II)** are defined like this:

1. If $f(x)$ continuous on $(a,b]$ and discontinuous at a, then
$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$
2. If $f(x)$ continuous on $[a,b)$ and discontinuous at b, then
$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$
3. If $f(x)$ discontinuous at c, where $a < c < b$, and continuous on $[a,c) \cup (c,b]$, then
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

In each case above, if the limit is finite, we say the improper integral _____________.
If the limit does not exist, the improper integral ________________.

Ex 4.
Evaluate:
$$\int_0^1 \frac{1}{1-x}\,dx$$

Ex 5.

Evaluate:

$$\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx$$

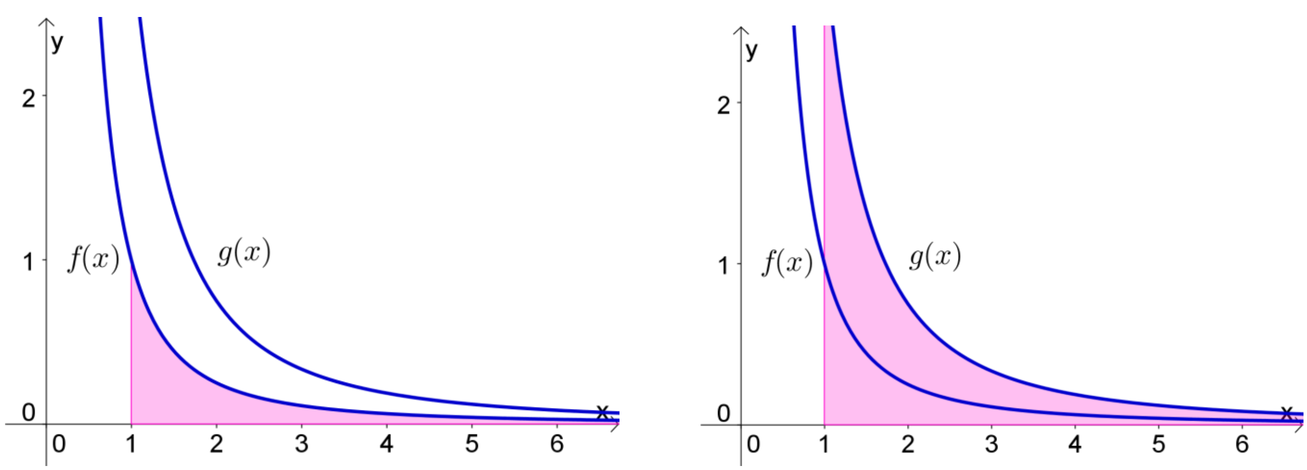
To determine if an integral converges or diverges, you can use the Comparison Test described below.

Comparison Test

Suppose f and g are continuous on $[a,\infty)$ and $0 \le f(x) \le g(x)$ in $[a,\infty)$.

If $\displaystyle\int_a^\infty g(x)\,dx$ converges, then $\displaystyle\int_a^\infty f(x)\,dx$ converges.

If $\displaystyle\int_a^\infty f(x)\,dx$ diverges, then $\displaystyle\int_a^\infty g(x)\,dx$ diverges.



Ex 6.

Does $\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ converge or diverge?

Ex 7.

Does $\int_1^\infty \frac{1}{\sqrt{x^2-0.1}}\,dx$ converge or diverge?

Practice

1. Evaluate:

$$\int_{-\infty}^{2} \frac{2\,dx}{x^2+4}$$

2. Evaluate:

$$\int_0^4 \frac{dx}{\sqrt{4-x}}$$

3. Does the following integral converge or diverge? (Use the Comparison Test.)

$$\int_4^\infty \frac{dx}{\sqrt{x}-1}$$

Q: What is it that you will break even when you name it?