1. Evaluate each integral.

a) $\int \frac{\sqrt{9-x^2}}{x^2}\,dx$

$x = 3\sin\theta \quad \left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right)$

$dx = 3\cos\theta\, d\theta$

$\sqrt{9-9\sin^2\theta} = \sqrt{9(1-\sin^2\theta)}$

$= \sqrt{9\cos^2\theta}$ Since $\cos\theta \ge 0$

$= 3\cos\theta$

$$= \int \frac{\sqrt{9-9\sin^2\theta}}{9\sin^2\theta} \cdot 3\cos\theta\, d\theta$$

$$= \int \frac{\cancel{3}\cos\theta}{\cancel{9}\sin^2\theta} \cdot \cancel{3}\cos\theta\, d\theta$$

$$= \int \frac{\cos^2\theta}{\sin^2\theta}\, d\theta$$

$$= \int \cot^2\theta\, d\theta$$

$$= \int (\csc^2\theta - 1)\, d\theta$$

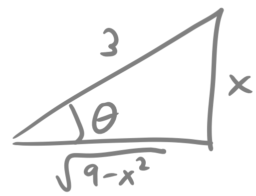
$$= -\cot\theta - \theta + C$$

$$= \boxed{-\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C}$$

$\sin\theta = \frac{x}{3}$

$\cot\theta = \frac{\sqrt{9-x^2}}{x}$

$\theta = \sin^{-1}\left(\frac{x}{3}\right)$



b) $\int \frac{dx}{\sqrt{x^2-a^2}}$ (Assume a is a positive constant.)

$x = a\sec\theta \quad \left(0 \le x < \frac{\pi}{2},\ \text{or}\ \pi \le x < \frac{3\pi}{2}\right)$

$dx = a\sec\theta\tan\theta\, d\theta$

$\sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2(\sec^2\theta - 1)}$

$= \sqrt{a^2\tan^2\theta}$

$= a\tan\theta$

$$= \int \frac{a\sec\theta\tan\theta}{\sqrt{a^2\sec^2\theta - a^2}}\, d\theta$$

$$= \int \frac{\cancel{a}\sec\theta\cancel{\tan\theta}}{\cancel{a}\cancel{\tan\theta}}\, d\theta$$

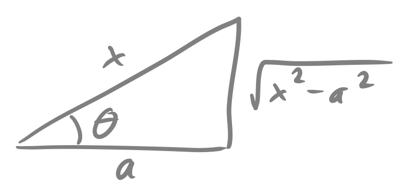
$$= \int \sec\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \boxed{\ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + C}$$

$\sec\theta = \frac{x}{a}$

$\tan\theta = \frac{\sqrt{x^2-a^2}}{a}$



c) $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$

$x = 2\tan\theta \qquad \left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$

$dx = 2\sec^2\theta\, d\theta$

$$= \int \frac{2\sec^2\theta}{4\tan^2\theta\sqrt{4\tan^2\theta+4}}\,d\theta$$

$\sqrt{4\tan^2\theta+4} = \sqrt{4(\tan^2\theta+1)}$

$= \sqrt{4\sec^2\theta}$

$= 2\sec\theta$

$$= \int \frac{\cancel{2}\sec^{\cancel{2}}\theta}{4\tan^2\theta\cdot\cancel{2\sec\theta}}\,d\theta$$

$$= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta$$

$$= \frac{1}{4}\int \frac{\left(\frac{1}{\cos\theta}\right)}{\left(\frac{\sin^2\theta}{\cos^2\theta}\right)}\,d\theta$$

$\frac{\left(\frac{1}{\cos\theta}\right)}{\left(\frac{\sin^2\theta}{\cos^2\theta}\right)} = \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta} = \frac{\cos\theta}{\sin^2\theta}$

$$= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

$u = \sin\theta$

$du = \cos\theta\, d\theta$

$$= \frac{1}{4}\int \frac{1}{u^2}\,du$$

$$= \frac{1}{4}\left(-\frac{1}{u}\right) + C$$

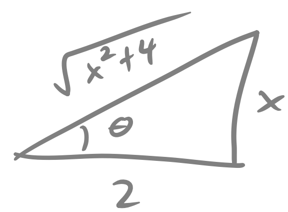
$u \to \theta$

$$= -\frac{1}{4\sin\theta} + C$$

$\theta \to x$

$\tan\theta = \frac{x}{2}$

$\sin\theta = \frac{x}{\sqrt{x^2+4}}$



$$= \boxed{-\frac{\sqrt{x^2+4}}{4x} + C}$$

Q: How can half of 12 be 7?