Power Series

A ____ is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

A ____ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots + c_n (x-a)^n + \cdots$$

(a is called the _______________ of the power series, and $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.)

Why are power series cool?

Power series will give us alternate forms of elementary functions (like $\sin x$, $\ln x$, e^x, etc.), and will allow us to create new types of functions. Power series will also be useful for approximating functions, and for helping us to integrate and solve differential equations that we previously couldn't.

Ex 1.

$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$ is a power series. For what values of x does it converge/diverge?

Ex 2.

For what value(s) of x does $\sum_{n=0}^{\infty} n!\, x^n$ converge?

Ex 3.

For what values of x does $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converge?

Ex 4.

For what values of x does $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$ converge?

Theorem:

$\sum_{n=0}^{\infty} c_n(x-a)^n$ either…

1. …converges only when $x = a$.
2. …converges absolutely for all x.
3. …converges absolutely for $|x-a| < R$ and diverges for $|x-a| > R$ for some $R > 0$. (Need to also check if converge when $|x-a| = R$.)

R is called the ____. For case 1, $R = 0$. For case 2, $R = \infty$.

Ex 5.

Find the radius and interval of convergence of $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.

Summary of How to Test Power Series for Convergence

1. Use Ratio Test (or Root Test) to find the radius of convergence.
2. If $R = 0$, the interval of convergence is $x = a$ (the center), and the series diverges for $x \neq a$.
3. If $R = \infty$, the interval of convergence is $(-\infty, \infty)$, and the series never diverges.
4. Otherwise, you know that the series converges absolutely on $|x-a| < R$ and diverges on $|x-a| > R$. Test for convergence at endpoints $x = a - R$ and $x = a + R$ (usually using Comparison Tests, Integral Test, or Alternating Series Test).

Practice

1. Find the radius and interval of convergence of $\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}$.

2. Find the radius and interval of convergence of $\sum_{n=1}^{\infty} (\ln n) x^n$.

3. Find the radius and interval of convergence of $\sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$.

Q: There is a word in the English language in which the first two letters signify a male, the first three letters signify a female, the first four signify a great man, and the whole word, a great woman. What is the word?