

## Absolute Convergence and the Ratio and Root Tests

$\sum a_n$  \_\_\_\_\_ if  $\sum |a_n|$  converges.

### The Absolute Convergence Test

If  $\sum |a_n|$  converges, then  $\sum a_n$  converges.

#### Ex 1.

Does  $\sum_{n=0}^{\infty} (-1)^n \frac{1}{2^n}$  converge or diverge?

#### Ex 2.

Does  $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$  converge or diverge?

**Note:** If  $\sum a_n$  converges, must  $\sum |a_n|$  converge? \_\_\_\_\_

For example,  $\sum (-1)^n \frac{1}{n}$  converges, but  $\sum \frac{1}{n}$  diverges.

$\sum a_n$  \_\_\_\_\_ if  $\sum a_n$  converges and  $\sum |a_n|$  diverges.

#### Ex 3.

For what values of  $p > 0$  does  $\sum \frac{(-1)^{n-1}}{n^p}$  converge conditionally and converge absolutely?

Remember how geometric series  $\sum ar^{n-1}$  converges when  $|r| < 1$ ?

In general for a series with positive terms, if (as  $n \rightarrow \infty$ ) the *next* term is a *fraction* ( $< 1$ ) of the *previous* term, then the series will converge. This is what the Ratio Test says.

### The Ratio Test

Suppose that

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

If  $L < 1$ , then  $\sum a_n$  converges absolutely.

If  $L > 1$  (or  $L$  infinite), then  $\sum a_n$  diverges.

If  $L = 1$ , then test inconclusive (try something else).

### Ex 4.

Does  $\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n}$  converge or diverge?

### Ex 5.

Does  $\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$  converge or diverge?

Suppose you rewrite a series  $\sum a_n$  as  $\sum (\sqrt[n]{a_n})^n$  and suppose that  $\sqrt[n]{a_n} \rightarrow \frac{2}{3}$ . Then  $\sum (\sqrt[n]{a_n})^n$  acts more and more like a geometric series ( $r \approx \frac{2}{3}$ ) as  $n$  gets large, which converges. This is the idea of the Root Test.

**The Root Test**

Suppose that  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L$

If  $L < 1$ , then  $\sum a_n$  converges absolutely.

If  $L > 1$  (or  $L$  infinite), then  $\sum a_n$  diverges.

If  $L = 1$ , then test inconclusive (try something else).

**Ex 6.**

Does  $\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$  converge or diverge?

**Ex 7.**

Does  $\sum_{n=1}^{\infty} \left(\ln\left(e^2 - \frac{1}{n}\right)\right)^{n+1}$  converge or diverge?

**Notes:**

- The Ratio Test is typically good with factorials.
- The Root Test works best when the terms are raised to the  $n$ th power.

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**Practice**

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1. Does  $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{n \ln n}$  converge absolutely, converge conditionally, or diverge?

2. Does  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} (n!)^2}{(2n)!}$  converge absolutely, converge conditionally, or diverge?

3. Does  $\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+2)^2}$  converge or diverge?

4. Does  $\sum_{n=1}^{\infty} \frac{4^n}{(3n)^n}$  converge or diverge?

5. Does  $\sum_{n=1}^{\infty} a_n$  converge or diverge, if  $\{a_n\}$  is defined by  $a_1 = 1$ ,  $a_{n+1} = \frac{1+\tan^{-1}n}{n} a_n$ ? (Hint: the Ratio Test is good with recursively-defined terms.)

6. Does  $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$  converge or diverge?

7. Does  $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$  converge or diverge?

**Challenge:** If  $a_n = \begin{cases} n/2^n & \text{if } n \text{ is a prime number} \\ 1/2^n & \text{otherwise} \end{cases}$ , does  $\sum_{n=1}^{\infty} a_n$  converge or diverge?

Q: What goes up and down but doesn't move?