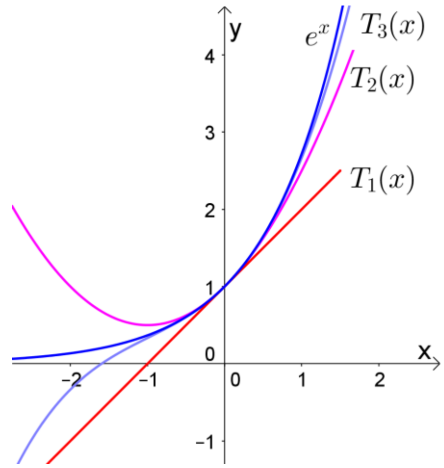
Applications of Taylor Polynomials



The partial sums of a Taylor series created at $x = a$ are written

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$T_n(x)$ is also called the **n^{th}-degree Taylor polynomial of f at a**.

For $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$,

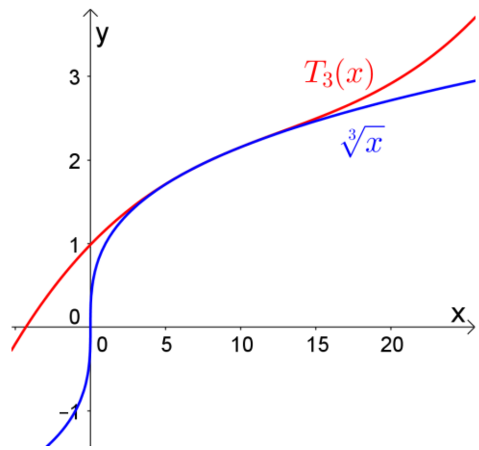
$T_1(x) = 1 + \frac{x}{1!}$

$T_3(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}$

Side note: $T_1(x)$ at $x = 0$ is the same as the tangent line at $x = 0$. In other words, it's the same as the good old linearization $L(x) = f(a) + f'(a)(x-a)$ from Math 180.

Ex 1.

Find the Taylor polynomial $T_3(x)$ for $f(x) = \sqrt[3]{x}$ centered at $a = 8$.



How good are Taylor polynomials at approximating Taylor series?

We can use Taylor's Inequality, described below, to put a bound on the difference between the Taylor polynomial values and the Taylor series values.

Taylor's Inequality

If $\left|f^{(n+1)}(x)\right| \le M$ for $|x-a| \le d$, then $|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$ for $|x-a| \le d$.

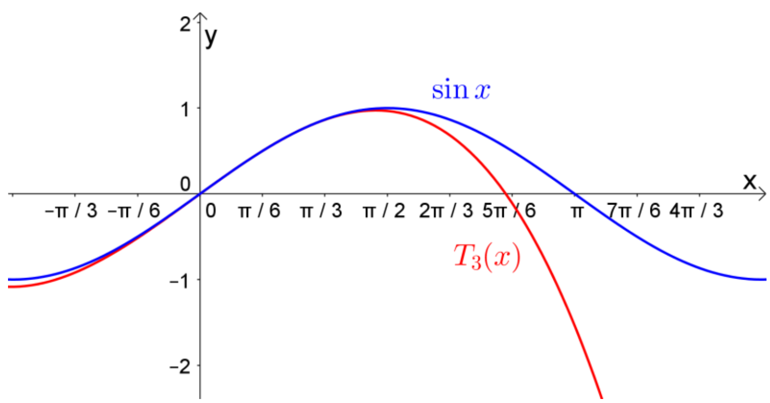
Here, $f(x) = T_n(x) + R_n(x)$, where $R_n(x)$ is the rest of the Taylor series (called the remainder).

Ex 2.

How accurate is the approximation in Ex 1 when $7 \le x \le 9$?

Practice

1. Find the Taylor polynomial $T_3(x)$ for $f(x) = \sin x$ centered at $a = \frac{\pi}{6}$. Then use Taylor's Inequality to estimate the accuracy of the $T_3(x)$ when x lies in the interval $0 \le x \le \frac{\pi}{3}$.



Joke: What did the buffalo say to his son when he left for college? Bison!

Bonus Joke: Did you hear the joke about the roof? Never mind, it's over your head!