Final Exam Review Exercise Answers – Math 181

1. 2 (set up: $\int_{\pi/4}^{3\pi/4}(\sin^2 x - \cos^2 x)\,dx + \int_{3\pi/4}^{5\pi/4}(\cos^2 x - \sin^2 x)\,dx$)

2. $\frac{8}{15}$ (set up: $\int_0^2 \frac{(2x-x^2)^2}{2}\,dx$)

3. Using washers: $\int_0^1 \left[\pi(2-(-1))^2 - \pi\left(2\sqrt{x}-(-1)\right)^2\right]dx = \frac{10\pi}{3}$

   Using shells: $\int_0^2 2\pi(y-(-1))\left(\frac{y^2}{4}-0\right)dy = \frac{10\pi}{3}$

4. Using shells: $\int_0^1 2\pi(1-y)\big((y-y^3)-0\big)\,dy = \frac{7\pi}{30}$

5. 80 ft-lb (set up: $\int_0^4 10x\,dx$)

6. 514.5 J (set up: $\int_0^5 3\cdot 9.8\cdot y\,dy + 3\cdot 9.8\cdot 5$)

7. 58643.1 ft-lb (set up: $\int_0^8 \pi\left(\frac{y}{2}\right)^2\cdot 62.5\cdot(13-y)\,dy$)

8. $2-\frac{4}{\pi}$ (set up: $\frac{1}{\frac{\pi}{2}-0}\int_0^{\pi/2} x^2\sin x\,dx$, use integration by parts)

9.
   a. 1 (Use integration by parts with $u=\cos^{-1}x$ and $dv=dx$)
   b. $\frac{\tan^3 x}{3}+\frac{\tan^5 x}{5}+C$ (Save one $\sec^2 x$ with dx, convert other $\sec^2 x$ to $1+\tan^2 x$, let $u=\tan x$)
   c. $\frac{x}{4\sqrt{4-x^2}}+C$ (Trig substitution with $x=2\sin\theta$)
   d. $\frac{(4+x^2)^{3/2}}{3}-4\sqrt{4+x^2}+C$
   e. $9x+2\ln|x|+\frac{1}{x}+7\ln|x-1|+C$
   f. $\ln(x^2+1)+\tan^{-1}x-2\ln|x-1|-\frac{1}{x-1}+C$ (Note: $\frac{-2x+4}{(x^2+1)(x-1)^2}=\frac{2x+1}{x^2+1}-\frac{2}{x-1}+\frac{1}{(x-1)^2}$)
   g. Diverges (first step looks like this: $\lim_{t\to 0^-}\int_{-1}^t \frac{1}{x^2}\,dx + \lim_{t\to 0^+}\int_t^2 \frac{1}{x^2}\,dx$)
   h. Converges to 0 (first step looks like this: $\lim_{t\to-\infty}\int_t^0 2xe^{-x^2}\,dx + \lim_{t\to\infty}\int_0^t 2xe^{-x^2}\,dx$)
   i. $3\sqrt[3]{x^2}e^{\sqrt[3]{x}} - 6\sqrt[3]{x}e^{\sqrt[3]{x}} + 6e^{\sqrt[3]{x}}+C$ (substitution with $u=\sqrt[3]{x}$, $3u^2du=dx$)
   j. $\frac{1}{2}\left(\ln\left|\sqrt{x^2+1}-1\right| - \ln\left|\sqrt{x^2+1}+1\right|\right)+C$ (substitution with $u=\sqrt{x^2+1}$)

      or $\ln\left|\frac{x}{\sqrt{x^2+1}+1}\right|+C$ (trig substitution with $x=\tan\theta$)
   k. $x-\ln|e^x+1|+C$ (substitution with $u=e^x+1$)

10. $\frac{x+3}{\sqrt{4x^4-3x-1}} > \frac{x}{\sqrt{4x^4}} = \frac{x}{2x^2} = \frac{1}{2x}$ so since $\int_2^\infty \frac{1}{2x}\,dx$ diverges, $\int_2^\infty \frac{x+3}{\sqrt{4x^4-3x-1}}\,dx$ diverges by comparison.

11. $\frac{595}{144}$ (set up: $\int_2^3 \sqrt{1+\left(\frac{x^3}{4}-\frac{1}{x^3}\right)^2}\,dx$)

12. 8π (set up: $\int_{-1}^1 2\pi\sqrt{4-x^2}\sqrt{1+\left(-\frac{x}{\sqrt{4-x^2}}\right)^2}\,dx$)

13. 246960 J (set up with a coordinate system that has x-axis along bottom of trapezoid and y-axis cutting trapezoid in half: $P_i = \rho g d = 1000 \cdot 9.8 \cdot (3 - y_i)$ and $A_i = 2 \cdot \left(\frac{15-y_i}{5}\right) \Delta y$; $\sum_{i=1}^{n} 1000 \cdot 9.8 \cdot (3 - y_i) \cdot 2 \cdot \left(\frac{15-y_i}{5}\right) \Delta y$; integral set up: $\int_0^3 1000 \cdot 9.8 \cdot (3 - y) \cdot 2 \cdot \left(\frac{15-y}{5}\right) dy$)

14. $\left(1, \frac{3}{5}\right)$ (set up: $A = \int_0^2 \left(x - (x^2 - x)\right) dx = \frac{4}{3}$, $\bar{x} = \frac{1}{A}\int_0^2 x\left(x - (x^2 - x)\right) dx$, $\bar{y} = \frac{1}{A}\int_0^2 \frac{x^2}{2}\left(x - (x^2 - x)\right) dx$)

15. $y'' + 3y' + y = (-t \sin t + 2 \cos t) + 3(t \cos t + \sin t) + t \sin t = 3t \cos t + 3 \sin t + 2 \cos t$;
$y(0) = 0 \cdot \sin 0 = 0$; $y'(0) = 0 \cdot \cos 0 + \sin 0 = 0$

16.
- a. $1 < y < 2$
- b. $y < 1$ or $y > 2$
- c. $y = 1$, $y = 2$

17.
- a. $y = e^{-xe^{-x} - e^{-x} + 1}$ (set up: separate variables $\int \frac{1}{y} dy = \int xe^{-x} dx$)
- b. $y = \frac{1}{2} e^{2 - \cos x}$ (set up: separate variables $\int \frac{1}{y} dy = \int \sin x \, dx$)

18. $\ln|y| - \frac{y^2}{2} = \frac{x^2}{2} + C$

19. 9.93% sugar (set up: $\frac{dA}{dt} = 2 - \left(\frac{A}{100}\right) \cdot 20$; $A(t) = 10 \pm Ke^{-t/5}$; $K = -4$; $A(t) = 10 - 4e^{-t/5}$)

20. $y - 1 = \frac{4}{\pi}\left(x - \frac{\pi^2}{16}\right)$ (or $y = \frac{4}{\pi}x - \frac{\pi}{4} + 1$)

21. $\sqrt{1+e^2} + \frac{1}{2}\ln\left(\frac{\sqrt{1+e^2}-1}{\sqrt{1+e^2}+1}\right) - \sqrt{2} - \frac{1}{2}\ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)$ (set up: $\int_0^1 \sqrt{1+e^{2t}} \, dt$, substitution with $u = \sqrt{1+e^{2t}}$; you'll get $\int_{\sqrt{2}}^{\sqrt{1+e^2}} \frac{u^2}{u^2-1} du$; divide, then do a partial fraction decomposition)

22.
- a. 4π (set up: $2\int_0^{\pi/2} \frac{1}{2}(4\cos\theta)^2 \, d\theta$; or it's just the area of a circle with radius 2!)
- b. $\frac{\pi}{4}$ (set up: $2\left(\int_0^{\pi/6} \frac{1}{2}(1 + \sin\theta)^2 \, d\theta - \int_0^{\pi/6} \frac{1}{2}(3\sin\theta)^2 \, d\theta + \int_{3\pi/2}^{2\pi} \frac{1}{2}(1 + \sin\theta)^2 \, d\theta\right)$)
- c. $\frac{\pi}{12}$ (set up: $2\int_0^{\pi/6} \frac{1}{2}(\cos 3\theta)^2 \, d\theta$)

23. 2 (set up: $\int_0^{\pi} \sqrt{\left(\sin^2\frac{\theta}{2}\right)^2 + \left(\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^2} \, d\theta$)

24. $\frac{3}{5\sqrt{3}}$

25. Horizontal: $\theta = \frac{3\pi}{4}$ and $\theta = \frac{7\pi}{4}$; Vertical: $\theta = \frac{\pi}{4}$ and $\theta = \frac{5\pi}{4}$ ($\frac{dy}{dx} = \frac{e^\theta \sin\theta + e^\theta \cos\theta}{e^\theta \cos\theta - e^\theta \sin\theta}$)

26.
- a. converges to e^{π}

b. diverges
c. converges to 0
d. diverges

27. $\frac{3}{2}$

28.

a. converges to $\frac{1}{e^4-e^2}$ (geometric series with $a=\frac{1}{e^4}$ and $r=\frac{1}{e^2}$)
b. converges (by Alternating Series Test)
c. diverges (by Integral Test)
d. diverges (by Test for Divergence)
e. converges to $\frac{5}{6}$ (telescoping sums, note that $\frac{2}{(n+1)(n+3)}=\frac{1}{n+1}-\frac{1}{n+3}$)
f. converges (by Root Test or Ratio Test)
g. diverges (by Limit Comparison Test with $\sum\frac{1}{2\sqrt{n}}$)
h. converges (by Comparison Test with $\sum\frac{2}{n^{3/2}}$)

29.

a. converges (by Ratio Test where $L=\frac{2}{3}<1$)
b. converges (by Root Test $L=0<1$)
c. converges conditionally ($\sum\left|(-1)^n\frac{3}{n+1}\right|$ diverges by Limit Comparison Test with $\sum\frac{1}{n}$ and $\sum(-1)^n\frac{3}{n+1}$ converges by Alternating Series Test)
d. converges absolutely ($\sum_{n=1}^{\infty}\left|\frac{(-2)^{n+1}}{n+5^n}\right|=\sum_{n=1}^{\infty}\frac{2^{n+1}}{n+5^n}$ converges by Comparison Test with $\sum 2\cdot\left(\frac{2}{5}\right)^n$)

30.

a. Interval: $(-2,0]$, Radius: 1 (Note: diverges at $x=-2$ by Limit Comparison test with $\sum\frac{1}{n}$, converges at $x=0$ by Alternating Series Test)
b. Interval: $(-\infty,\infty)$, Radius: ∞

31. $\sum_{n=0}^{\infty}\left(-\frac{1}{2^{n+1}}x^{n+2}\right)$ and interval of convergence $(-2,2)$ (or reindex to get $\sum_{n=2}^{\infty}\left(-\frac{1}{2^{n-1}}x^n\right)$)

32. $\ln 4-\sum_{n=1}^{\infty}\frac{1}{n\cdot 4^n}x^n$ and radius of convergence is 4 (Hint: $\ln(4-x)=\ln 4\left(1+\left(-\frac{x}{4}\right)\right)=\ln 4+\ln\left(1+\left(-\frac{x}{4}\right)\right)$)

33. $\sum_{n=0}^{\infty}\frac{2(\ln 2)^n}{n!}(x-1)^n$ and radius of convergence is ∞

34. $1+\frac{1}{2}(x-1)+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}\cdot 1\cdot 3\cdot 5\cdots(2n-3)}{2^n n!}(x-1)^n$ and radius of convergence is 1

35. $\sum_{n=0}^{\infty}\frac{2^n}{n!}x^{2n+1}$ and radius of convergence is ∞

36. $\frac{1}{2}+\sum_{n=1}^{\infty}\frac{1\cdot 4\cdot 7\cdot\ldots\cdot(3n-2)}{2\cdot n!\cdot 24^n}x^n$ and radius of convergence is 8

37. $C+\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!(4n+3)}x^{4n+3}$

38. 0.003 (Note: the antiderivative is $\frac{x}{3} - \frac{x^5}{5} + \frac{x^7}{14} - \frac{x^9}{54} + \cdots$)

39. -1 (Note: at some point you'll get $\lim_{x\to\infty}\left(-1 + \frac{1}{2x^2} - \frac{1}{6x^4} + \frac{1}{24x^6} - \cdots\right)$)

40. $x^2 - \frac{2}{3}x^4 + \frac{23}{45}x^6 + \cdots$ (Note: $(\tan^{-1} x)^2 = (\tan^{-1} x)(\tan^{-1} x)$)

41. $x + \frac{1}{2}x^2 + \frac{5}{6}x^3 + \cdots$ (Note: $\frac{\ln(1+x)}{1-x} = \ln(1+x) \cdot \frac{1}{1-x}$; Or, you can do long division…)

42. $T_3(x) = x - \frac{x^3}{2!} + \frac{x^5}{4!}$

43. $T_3(x) = (x-1) + \frac{(x-1)^2}{2} - \frac{(x-1)^3}{6}$; $T_3(x)$ is accurate within 0.041667 as long as $0.5 \le x \le 1.5$ (Note: $f^{(4)}(x) = \frac{2}{x^3}$; use $M = 16$)