Areas of Surfaces of Revolution

Suppose we take an arc and spin it about x-axis.

It turns out that the area of the surface that's generated can be calculated by: $\boldsymbol{\int_a^b 2\pi r\, ds}$

r is the distance from the x-axis to the curve, which we can represent with y.

ds is the arc length of an infinitesimally small piece of the curve, where you can either use

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \quad \text{or} \quad ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Note: We need $f(x) \geq 0$ on $[a, b]$ and f' is continuous on $[a, b]$.

So we have: $\boldsymbol{S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx}$

Ex 1.

Find the area of the surface generated by revolving the curve $y = \sqrt{x}, \frac{3}{4} \leq x \leq \frac{15}{4}$ about the x-axis.

If you rotate about the y-axis, you can still use $\int_a^b 2\pi r\, ds$.
But this time, r is the distance from the y-axis to the curve, which we can represent by x.

Ex 2.
Find the area of the surface generated by revolving the curve $x = 1 - y$, $0 \leq y \leq 1$ about the y-axis.

Ex 3.
Find the area of the surface generated by revolving the curve $x = 1 + 2y^2$, $1 \leq y \leq 2$ about the x-axis.

Practice

1. Find the surface area of a sphere with radius 3 using calculus. (Hint: Spin a semi-circle with radius 3 about the x-axis. Remember that the equation for the relevant circle here is $x^2 + y^2 = 3^2$.) You can check that you got the answer correct by using the formula for the surface area of a sphere, $S = 4\pi r^2$.

Unrelated Joke: There are 10 kinds of people in the world: those who understand binary, and those who don't.