Approximate Integration

Some functions, like e^{x^2}, do not have elementary antiderivatives. So, what do we do when we need to compute $\int_0^1 e^{x^2}\,dx$? We use numerical integration techniques to approximate the answer.

We have already used rectangles to approximate the area under a curve, but here are two other ways to get good approximations faster.

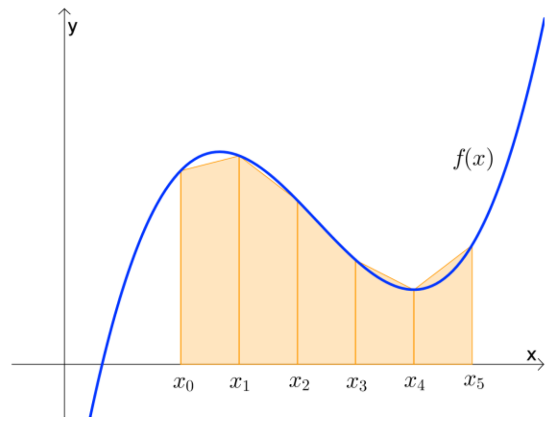
Trapezoids

If we use n trapezoids to approximate the area under a curve, we get the **Trapezoidal Rule**:

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

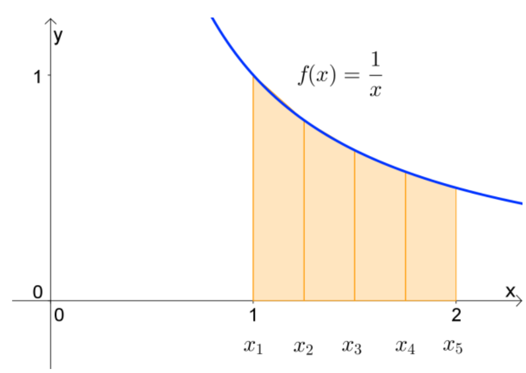
Here, $x_0 = a$, $x_n = b$, and Δx is the width of each subinterval.

Note: $\Delta x = \frac{b-a}{n}$



Ex 1.

Use the Trapezoidal Rule with $n = 4$ subintervals to approximate $\int_1^2 \frac{1}{x}\,dx$.



Simpson's Rule (Parabolas)

It turns out that using parabolas is another way to get good approximations quickly (see applet).
Simpson's Rule is the resulting approximation formula:

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

Note: To use Simpson's Rule, n must be even.

Ex 2.

Use Simpson's Rule with $n = 4$ subintervals to approximate $\int_1^2 \frac{1}{x}\,dx$.

How close do the Trapezoidal Rule and Simpson's Rule get to the actual areas? It turns out that we can put bounds on the error between the estimate area and actual area $\left(\int_a^b f(x)\,dx\right)$.

Trapezoidal Rule: $|\boldsymbol{E_T}| \leq \frac{\boldsymbol{M(b-a)^3}}{\boldsymbol{12n^2}}$ (M is any upper bound of $|f''|$ on $[a, b]$, and n is # of subintervals)

Simpson's Rule: $|\boldsymbol{E_S}| \leq \frac{\boldsymbol{M(b-a)^5}}{\boldsymbol{180n^4}}$ (M is any upper bound of $\left|f^{(4)}\right|$ on $[a, b]$, and n is # of subintervals)

Ex 3.
Estimate the error in the approximation of the integral from Ex 1 (which used the Trapezoidal Rule).

Ex 4.
Estimate the error in the approximation of the integral from Ex 2 (which used Simpson's Rule).

Ex 5.

How large do we have to choose n so that the approximation to the integral in Ex 2 (which used Simpson's Rule) is accurate to within 0.0001?

Notes:

"Find T_8" means "use the Trapezoidal Rule with $n = 8$".

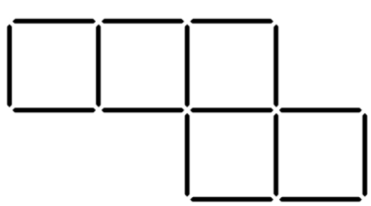
"Find S_6" means "use Simpson's Rule with $n = 6$".

Practice

1. Use the Trapezoidal Rule with $n = 4$ steps to approximate $\int_{-1}^{1}(x^2 - 1)dx$ and estimate the error in the approximation (i.e. find an upper bound for $|E_T|$). Evaluate the integral directly and find $|E_T|$.

Q: What gets wet when drying?

Bonus puzzle!



Move 2 line segments to make 4 squares.
One square will be larger than the other 3.