

Integration of Rational Functions by Partial Fractions

Let's review the **method of partial fractions**, which is a technique to break rational functions up into the sum of simpler rational functions.

Example:
$$\frac{5x-3}{x^2-2x-3} = \frac{5x-3}{(x+1)(x-3)} = \frac{2}{x+1} + \frac{3}{x-3}$$

Ex 1.

Expand the quotient by partial fractions.

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)}$$

Ex 2.

Evaluate the following integral.

$$\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)} dx$$

In general, we need to factor the denominator first. What happens if we don't get linear factors?

1. If we get a **repeated linear factor** $(x - r)^m$, then we'll have corresponding partial fractions:

$$\frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \cdots + \frac{A_m}{(x-r)^m}$$

2. If we get an **irreducible quadratic factor** $x^2 + px + q$, then we'll have a corresponding partial fraction: $\frac{Ax+B}{x^2+px+q}$

3. If we get a **repeated irreducible quadratic factor** $(x^2 + px + q)^n$, then we'll have corresponding partial fractions: $\frac{B_1x+C_1}{(x^2+px+q)} + \frac{B_2x+C_2}{(x^2+px+q)^2} + \cdots + \frac{B_nx+C_n}{(x^2+px+q)^n}$

Ex 3.

$$\int \frac{6x+7}{(x+2)^2} dx$$

Note: If degree of top polynomial is ___ degree of bottom polynomial, then _____ first.

Ex 4.

$$\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} dx$$

Ex 5.

$$\int \frac{-2x+4}{(x^2+1)(x-1)^2} dx$$

Practice

1. Evaluate each integral.

a) $\int \frac{x^2+8}{x^2-5x+6} dx$

b) $\int \frac{1}{x(x^2+1)^2} dx$ (Hint: For part b, it's probably easier to expand and match coefficients rather than plug in x -values. This often helps with irreducible quadratic factors.)

Q: What do you get when you expand $(x - a)(x - b)(x - c)\dots(x - y)(x - z)$?