The Comparison Tests

Remember the Comparison Test for integrals? Series have a similar test to determine convergence/divergence. But first, it will help to have a bunch of series to compare to.

$$n^n \gg n! \gg \cdots \gg 3^n \gg 2^n \gg \cdots \gg n^2 \gg n^{1.1} \gg \cdots \gg n \gg \sqrt{n} \gg \sqrt[3]{n} \gg \cdots \gg \ln n$$

$$\frac{1}{n^n} \ll \frac{1}{n!} \ll \cdots \ll \frac{1}{3^n} \ll \frac{1}{2^n} \ll \cdots \ll \frac{1}{n^2} \ll \frac{1}{n^{1.1}} \ll \cdots \ll \frac{1}{n} \ll \frac{1}{\sqrt{n}} \ll \frac{1}{\sqrt[3]{n}} \ll \cdots \ll \frac{1}{\ln n}$$

How can we show, for example, that $2^n \gg n^3$? Limits and L'Hospital!

$$\lim_{n\to\infty} \frac{2^n}{n^3} = \lim_{n\to\infty} \frac{2^n \ln 2}{3n^2} = \lim_{n\to\infty} \frac{2^n (\ln 2)^2}{6n} = \lim_{n\to\infty} \frac{2^n (\ln 2)^3}{6} = \infty$$

So, as n gets really big, 2^n gets to be much bigger than n^3.

The Comparison Test

Suppose a_n and b_n have nonnegative terms, and N is some integer.

If $a_n \le b_n$ for all $n > N$ and if $\sum b_n$ converges, then the smaller $\sum a_n$ also converges.

If $b_n \le a_n$ for all $n > N$ and if $\sum b_n$ diverges, then the bigger $\sum a_n$ also diverges.

Ex 1.

Does $\sum_{n=1}^{\infty} \frac{5}{5n-1}$ converge or diverge?

Ex 2.

Does $\sum_{n=4}^{\infty} \frac{n-3}{2+n^2\sqrt{n}}$ converge or diverge?

Ex 3.

Does $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ converge or diverge?

What about $\sum \frac{1}{2^n-1}$?

When n is large, $\frac{1}{2^n-1}$ acts like $\frac{1}{2^n}$. Since $\sum \frac{1}{2^n}$ converges, $\sum \frac{1}{2^n-1}$ should converge, too.

We want to compare it to $\sum \frac{1}{2^n}$ but the problem is that $\frac{1}{2^n-1} \geq \frac{1}{2^n}$ (not "$\leq$").

What to do? Thankfully, the Limit Comparison Test comes to our rescue!

The Limit Comparison Test (LCT)

Suppose a_n and b_n have positive terms for all $n \geq N$ (N is some integer).

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Ex 4.

Does $\sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}$ converge or diverge?

Ex 5.

Does $\sum_{n=2}^{\infty} \frac{1+n\ln n}{n^2+5}$ converge or diverge?

Ex 6.

Does $\sum_{n=1}^{\infty} \frac{n^2+2n}{\sqrt{3+n^7}}$ converge or diverge?

Practice

1. Does $\sum_{n=8}^{\infty} \frac{1}{\sqrt[3]{n}-1}$ converge or diverge?

2. Does $\sum_{n=1}^{\infty} \frac{2^n}{3+4^n}$ converge or diverge?

3. Does $\sum_{n=1}^{\infty} \frac{n+3}{n^4-n^3+2n}$ converge or diverge?

Challenge: Does $\sum_{n=1}^{\infty} \frac{1}{1+2+3+\cdots+n}$ converge or diverge?

Q: What is the word that everybody always says wrong?