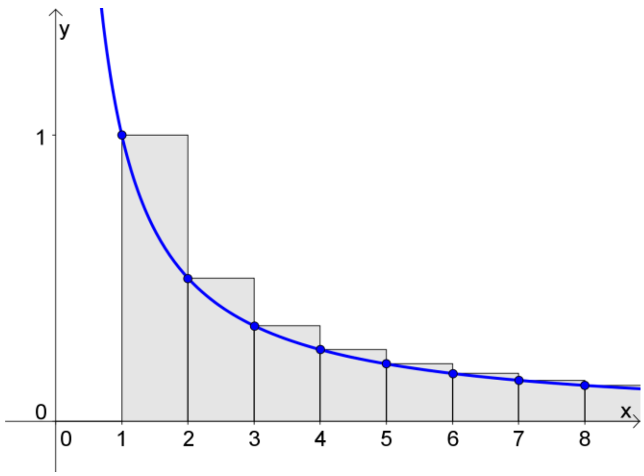
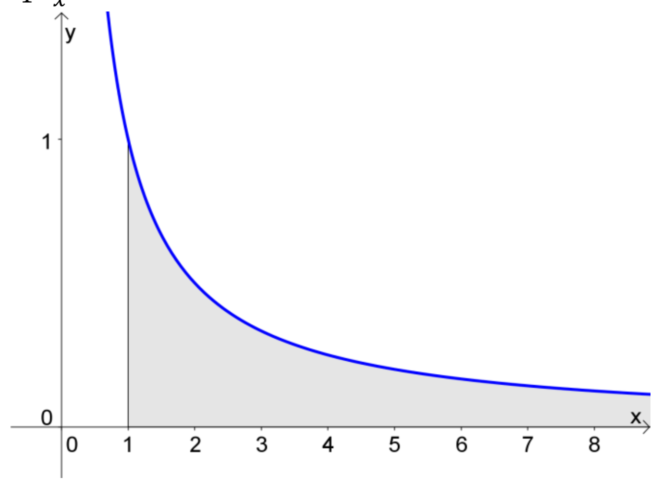
The Integral Test and Estimates of Sums

Let's look at the series $\sum_{n=1}^{\infty} \frac{1}{n}$ and compare it with $\int_1^{\infty} \frac{1}{x}\,dx$:

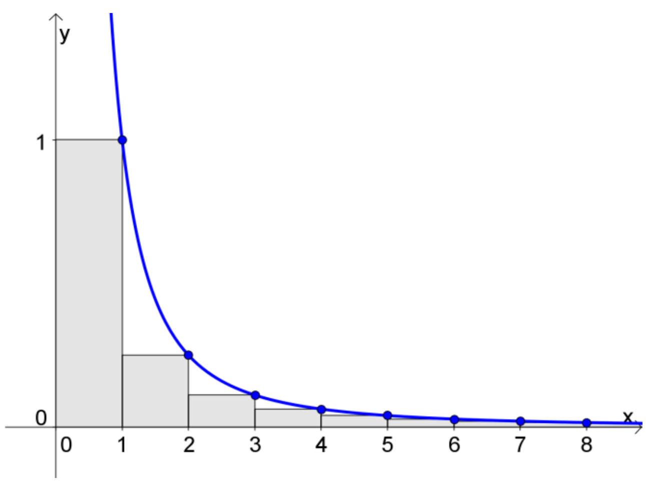


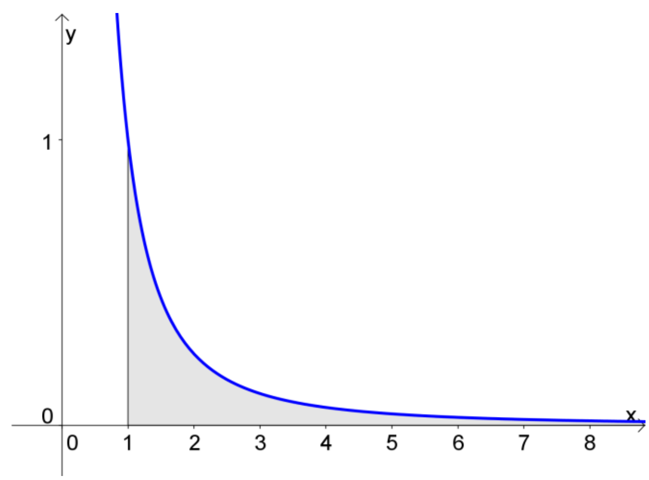


Note that $\int_1^{\infty} \frac{1}{x}\,dx < \sum_{n=1}^{\infty} \frac{1}{n}$.

So, by comparison, since $\int_1^{\infty} \frac{1}{x}\,dx$ _____________, then the larger $\sum_{n=1}^{\infty} \frac{1}{n}$ must also ____________.

Now let's look at the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and compare it with $\int_1^{\infty} \frac{1}{x^2}\,dx$:





Note that $\sum_{n=1}^{\infty} \frac{1}{n^2} < 1 + \int_1^{\infty} \frac{1}{x^2}\,dx$.

So, by comparison, since $\int_1^{\infty} \frac{1}{x^2}\,dx$ _____________, then the smaller $\sum_{n=1}^{\infty} \frac{1}{n^2}$ must also ____________.

The Integral Test (connects series with integrals)

Suppose that $a_n = f(n)$, where $f(x)$ is **continuous**, **positive**, and **decreasing** for all $x \geq N$ (N is some positive integer).

Then $\sum_{n=N}^{\infty} a_n$ and $\int_N^{\infty} f(x)\,dx$ both converge or both diverge.

Ex 1.

Does $\sum_{n=2}^{\infty} \frac{\ln n}{n}$ converge or diverge?

Ex 2.

Does $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converge or diverge?

Recall: $\int_1^\infty \frac{1}{x^p}\,dx$ converges if $p > 1$, and diverges if $p \le 1$.

Ex 3.

For what values of p does $\sum_{n=1}^\infty \frac{1}{n^p}$ converge or diverge? ($\sum_{n=1}^\infty \frac{1}{n^p}$ is called a **p-series**.)

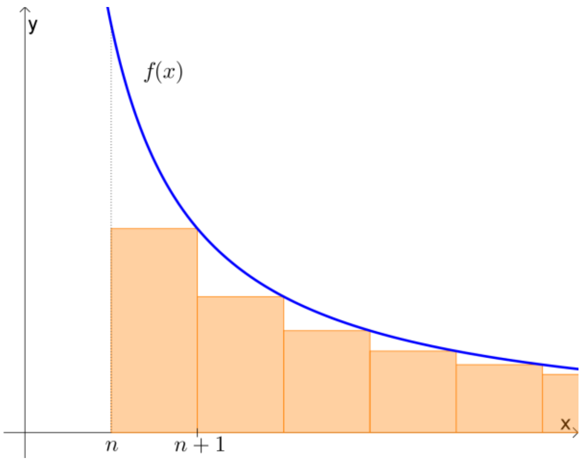
ex: $\sum_{n=1}^\infty \frac{1}{n^2}$ converges.

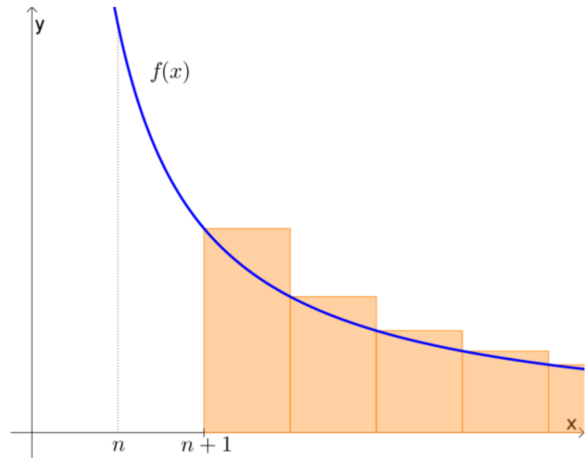
ex: $\sum_{n=1}^\infty \frac{1}{\sqrt{n}}$ diverges.

Estimates of Sums

How accurate is a given partial sum? In other words, how close is the sum of the first n terms (s_n) to the sum of the infinite number of terms ($\sum a_n$)?

$$\sum a_n = a_1 + a_2 + \cdots + a_n + a_{n+1} + a_{n+2} + \cdots$$





Remainder Estimate for the Integral Test

Suppose that $a_n = f(n)$, where $f(x)$ is continuous, positive, and decreasing for all $x \geq n$.
If $\sum a_n$ converges, then

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx$$

Ex 4.

Estimate the error in using s_{10} as an approximation to the sum of $\sum_{n=1}^{\infty} \frac{1}{n^3}$. Also, how many terms are needed to make sure that the sum is accurate to within 0.0005?

If we take

$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_{n}^{\infty} f(x)\,dx$$

and add s_n to all three parts, we get

$$s_n + \int_{n+1}^{\infty} f(x)\,dx \le s_n + R_n \le s_n + \int_{n}^{\infty} f(x)\,dx.$$

Since $s_n + R_n = \sum a_n$, we get an upper and lower bound for our series:

$$\boldsymbol{s_n + \int_{n+1}^{\infty} f(x)\,dx \le \sum a_n \le s_n + \int_{n}^{\infty} f(x)\,dx}$$

(Note: This is referred to as $\boxed{3}$ in the book.)

Ex 5.

Use $\boxed{3}$ with $n = 10$ to give an improved estimate of the sum of $\sum_{n=1}^{\infty} \frac{1}{n^3}$ (better than s_{10}).

Practice

1. Does $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ converge or diverge? (Use the Integral Test. Be sure to show why the terms are decreasing.)

Challenge: Is there a "smallest" divergent series? (That is, a divergent series $\sum a_n$ where the terms a_n are smaller than the terms of any other divergent series.)

Q: A man leaves home and, after making three left turns, he ends up back at home, and finds two masked men waiting for him. What is happening?