1. Write out the first few terms of the following series to show how the series starts. Then find the sum of the series.

$\sum_{n=0}^{\infty}(-1)^n \frac{5}{4^n}$ $= 5 - \frac{5}{4} + \frac{5}{16} - \ldots$

$\frac{a}{1-r}$

$a=5$ $\quad ar = -\frac{5}{4}$

$5r = -\frac{5}{4}$

$r = -\frac{1}{4}$

$$\text{Sum} = \frac{5}{1-(-\frac{1}{4})} = \frac{5}{5/4} = \boxed{4}$$

2. Rewrite $3.1\overline{23} = 3.12323232323 \ldots$ using a geometric series, and then express it as the ratio of two integers.

$$3.1232323\ldots = 3.1 + \underbrace{\frac{23}{10^3} + \frac{23}{10^5} + \frac{23}{10^7} + \ldots}$$

$a = \frac{23}{10^3}$, $r = \frac{1}{10^2}$

$$\text{Sum} = \frac{23/10^3}{1 - 1/10^2} = \frac{23/1000}{99/100} = \frac{23}{990}$$

$$= \frac{31}{10} + \frac{23}{990}$$

$$= \boxed{\frac{1546}{495}}$$

3. Does the following series converge or diverge? If it converges, find its sum.

$\sum_{n=1}^{\infty} \frac{n}{2n+5}$

$$\lim_{n\to\infty} \frac{n}{2n+5} = \lim_{n\to\infty} \frac{1}{2+\frac{5}{n}} = \frac{1}{2} \neq 0$$

So, $\sum_{n=1}^{\infty} \frac{n}{2n+5}$ $\boxed{\text{diverges}}$ (by Test for Divergence)

4. Does the following series converge or diverge? If it converges, find its sum. (Hint: Use log properties first to rewrite $\ln\frac{\sqrt{n}}{\sqrt{n+1}}$.)

$$\sum_{n=1}^{\infty}\ln\frac{\sqrt{n}}{\sqrt{n+1}} = \sum_{n=1}^{\infty}\left(\ln\sqrt{n}-\ln\sqrt{n+1}\right) = \sum_{n=1}^{\infty}\left(\tfrac{1}{2}\ln n-\tfrac{1}{2}\ln(n+1)\right)$$

$$S_n = \left(\underbrace{\tfrac{1}{2}\ln 1}_{0}-\cancel{\tfrac{1}{2}\ln 2}\right)+\left(\cancel{\tfrac{1}{2}\ln 2}-\cancel{\tfrac{1}{2}\ln 3}\right)+\left(\cancel{\tfrac{1}{2}\ln 3}-\cancel{\tfrac{1}{2}\ln 4}\right)+\ldots+\left(\cancel{\tfrac{1}{2}\ln n}-\tfrac{1}{2}\ln(n+1)\right)$$

$$= -\tfrac{1}{2}\ln(n+1)$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} -\tfrac{1}{2}\ln(n+1) = -\infty$$

Thus, $\sum_{n=1}^{\infty}\ln\frac{\sqrt{n}}{\sqrt{n+1}}$ $\boxed{\text{diverges}}$.

(Note that the series diverged even though $\ln\frac{\sqrt{n}}{\sqrt{n+1}}\to 0$ as $n\to\infty$.)

Challenge: Find the value for b for which $1 + e^b + e^{2b} + e^{3b} + \cdots = 9$.

Challenge: If $\sum a_n$ and $\sum b_n$ both diverge, must $\sum(a_n + b_n)$ diverge?

Q: What has many keys but can't open any doors?

Challenge:

$$1+e^b+e^{2b}+e^{3b}+\dots = \underbrace{1+e^b+(e^b)^2+(e^b)^3+\dots}_{\substack{\text{geometric Series} \\ a=1,\ r=e^b}}$$

We want

$$\frac{a}{1-r} = 9$$

$$\frac{1}{1-e^b} = 9 \quad \leftarrow \text{solve for } b$$

$$\frac{1}{9} = 1-e^b$$

$$e^b = \frac{8}{9}$$

$$\boxed{b = \ln\frac{8}{9}}$$

Challenge:

$\boxed{\text{No}}$

For example, $a_n = (-1)^n$ $\{-1, 1, -1, 1, \dots\}$

$b_n = (-1)^{n+1}$ $\{1, -1, 1, -1, \dots\}$

$a_n + b_n$ $\{0, 0, 0, 0, \dots\}$

Here, both $\sum a_n$ and $\sum b_n$ diverge, but $\sum(a_n+b_n)$ converges (to 0).