Series

A __________ is the ______ of the terms of a sequence.

Sequence: $a_1, \quad a_2, \quad a_3, \quad \ldots, \quad a_n, \quad \ldots$

Series: $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ $\qquad \left(\sum_{n=1}^{\infty} a_n\right)$

What does the following series add up to? $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots$ $\qquad \left(\sum_{n=1}^{\infty} \frac{1}{2^n}\right)$

Let's try adding the terms one at a time:

What we just calculated are called the ________________ of the series. In general, they look like:

$s_1 = a_1$

$s_2 = a_1 + a_2$

$s_3 = a_1 + a_2 + a_3$

$s_4 = a_1 + a_2 + a_3 + a_4$

…

$s_n = a_1 + a_2 + a_3 + a_4 + \cdots + a_n$ $\qquad (s_n = \sum_{k=1}^{n} a_k)$

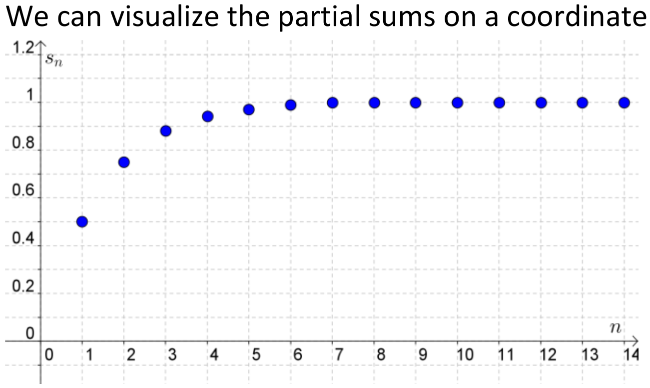
…

The partial sums $s_1, s_2, s_3, s_4, \ldots$ make a sequence $\{s_n\}$ that will either converge or diverge.

If $\{s_n\}$ converges to L, then we say $\sum_{n=1}^{\infty} a_n$ ______________ to L.

If $\{s_n\}$ diverges, then we say $\sum_{n=1}^{\infty} a_n$ ______________.

We can visualize the partial sums on a coordinate system, with n on the x-axis, and s_n on the y-axis.



$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$ is an example of a ________________________, which generally has the form:

$$\boldsymbol{a+ar+ar^2+ar^3+\cdots+ar^{n-1}+\cdots} \qquad \left(\sum_{\boldsymbol{n=1}}^{\infty}\boldsymbol{ar^{n-1}}\right)$$

Above, $a \neq 0$ is the first term, and r is called the ____________________.

When does a geometric series $\sum_{\boldsymbol{n=1}}^{\infty}\boldsymbol{ar^{n-1}}$ converge or diverge?

Ex 1.

Find the sum of $5-\frac{10}{3}+\frac{20}{9}-\frac{40}{27}+\cdots$

$$\sum_{n=1}^{\infty}ar^{n-1}=a+ar+ar^2+\cdots$$
converges to $\frac{a}{1-r}$ if $|r|<1$
diverges if $|r|\geq 1$

Ex 2.

Express $2.3\overline{17} = 2.317171717 \ldots$ as the ratio of two integers.

"Telescoping Sums"

Ex 3.

Find the sum of $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

Theorem: If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

ex: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges, so $\lim_{n\to\infty} \frac{1}{n(n+1)} = 0$.

Test for Divergence: If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Ex 4.

Does $\sum_{n=1}^{\infty} n^2$ converge or diverge?

Ex 5.

Does $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ converge or diverge?

Ex 6.

Does $\sum_{n=1}^{\infty} (-1)^{n+1}$ converge or diverge?

Note: *Caution!* Sometimes $\lim_{n\to\infty} a_n = 0$, but $\sum_{n=1}^{\infty} a_n$ still diverges!

ex: Let's show that even though $\lim_{n\to\infty} \frac{1}{n} = 0$, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges:

Notes:

$$\sum a_n = \sum_{n=1}^{\infty} a_n$$

If $\sum a_n$ and $\sum b_n$ converge, then

1. $\sum (a_n \pm b_n) = \sum a_n \pm \sum b_n$
2. $\sum k a_n = k \sum a_n$ (k is a constant)

Ex 7.

Does $\sum_{n=1}^{\infty} \frac{3^n + 2^n}{5^n}$ converge or diverge? If it converges, find its sum.

Ex 8.

Does $\sum_{n=1}^{\infty} \frac{4}{2^n}$ converge or diverge? If it converges, find its sum.

Practice

1. Write out the first few terms of the following series to show how the series starts. Then find the sum of the series.

   $\sum_{n=0}^{\infty}(-1)^n \frac{5}{4^n}$

2. Rewrite $3.1\overline{23} = 3.12323232323 \ldots$ using a geometric series, and then express it as the ratio of two integers.

3. Does the following series converge or diverge? If it converges, find its sum.

   $\sum_{n=1}^{\infty} \frac{n}{2n+5}$

4. Does the following series converge or diverge? If it converges, find its sum. (Hint: Use log properties first to rewrite $\ln \frac{\sqrt{n}}{\sqrt{n+1}}$.)

$\sum_{n=1}^{\infty} \ln \frac{\sqrt{n}}{\sqrt{n+1}}$

Challenge: Find the value for b for which $1 + e^b + e^{2b} + e^{3b} + \cdots = 9$.

Challenge: If $\sum a_n$ and $\sum b_n$ both diverge, must $\sum (a_n + b_n)$ diverge?

Q: What has many keys but can't open any doors?