Sequences

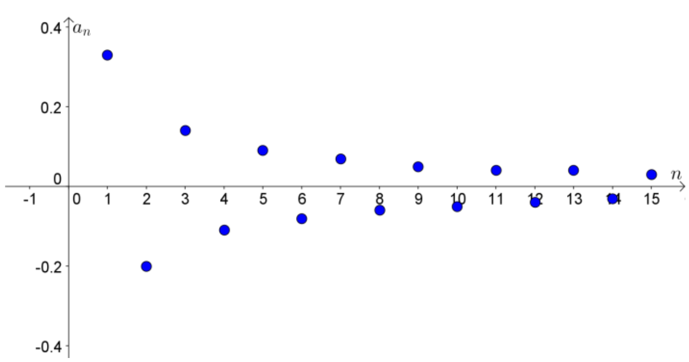
An **infinite sequence** is an ordered list of numbers $a_1, a_2, a_3, \dots, a_n, \dots$
ex: $3,\ 7,\ 11,\ 15,\ 19, \dots$

Other notations for a sequence $a_1, a_2, a_3, \dots$: $\{a_1, a_2, a_3, \dots\}$ $\{a_n\}$ $\{a_n\}_{n=1}^{\infty}$

ex: Suppose $a_n = \frac{n}{n+1}$ gives a formula for the sequence $\{a_n\}$. Then $\{a_n\} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots .\right\}$.

Ex 1.
Find a_1, a_2, a_3, and a_4 if $a_n = \frac{(-1)^{n+1}}{2n+1}$.

We can visualize sequences as a bunch of dots on a coordinate system, with n on the x-axis, and a_n on the y-axis.



In the above graph, it looks like $\lim_{n\to\infty} a_n =$ ____. Since the limit exists, we say that a_n __________. If the limit didn't exist, we'd say a_n ____________ (ex: $\{n^2\}$ diverges).

Definition: (formal)
$\{a_n\}$ **converges to L** if for every $\epsilon > 0$ there is an integer N such that for all $n > N$, $|a_n - L| < \epsilon$.

Ex 2.

Does $\left\{\frac{1}{n}\right\}$ converge or diverge?

Ex 3.

Does $\{1, -1, 1, -1, 1, -1, 1, -1, \ldots\}$ converge or diverge? Find a formula for the sequence.

Ex 4.

Find a formula for the nth terms of the sequence $1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, \ldots$

All the same limit rules that we know and love still apply. So, if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$,

$$\lim_{n\to\infty} (a_n \pm b_n) = A \pm B$$

$$\lim_{n\to\infty} (k \cdot b_n) = k \cdot B$$

$$\lim_{n\to\infty} (a_n \cdot b_n) = A \cdot B$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B} \qquad (\text{if } B \neq 0)$$

Ex 5.

Find the following limits.

$\lim_{n\to\infty} \left(-\frac{1}{n}\right) =$

$\lim_{n\to\infty} \left(\frac{n-1}{n}\right) =$

$\lim_{n\to\infty} \left(\frac{4-7n^6}{n^6+3}\right) =$

Ex 6.

Suppose $\{a_n\} = \{1, 2, 3, \ldots\}$ and $\{b_n\} = \{-1, -2, -3, \ldots\}$. Does $\{a_n + b_n\}$ converge or diverge?

The Squeeze Theorem for Sequences

If $a_n \le b_n \le c_n$ for all $n > N$, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ also.

Ex 7.

Do the following sequences converge or diverge? If they converge, what do they converge to?

$\left\{\frac{\cos n}{n}\right\}$

$\left\{(-1)^n \frac{1}{n}\right\}$

Theorem: (connects sequences and functions)

Suppose we have a sequence $\{a_n\}$ and a function $f(x)$ such that $a_n = f(n)$ for $n \ge n_0$. Then,

$\lim_{x\to\infty} f(x) = L \quad \Rightarrow \quad \lim_{n\to\infty} a_n = L$

Ex 8.

Does $\left\{\frac{\ln n}{n}\right\}$ converge or diverge? If it converges, what does it converge to?

Note: Often, we'll just treat n as a continuous real variable (rather than a natural number), and just differentiate directly with respect to n.

Ex 9.

Does the sequence $a_n = \left(\frac{n+1}{n-1}\right)^n$ converge or diverge? If it converges, what does it converge to?

Recursive Definitions

A recursively defined sequence is one where each term depends on one or more terms before it.

Ex 10.

Write the first four terms of the sequence defined below.

$a_1 = 1, \quad a_n = a_{n-1} + 2$

Ex 11.

Assume that the following sequence converges and find its limit.

$a_1 = -1, \quad a_{n+1} = \frac{a_n+6}{a_n+2}$

$\{a_n\}$ is **increasing** if $a_n < a_{n+1}$ for all n.
$\{a_n\}$ is **decreasing** if $a_n > a_{n+1}$ for all n.
$\{a_n\}$ is **monotonic** if it is either increasing or decreasing.

$\{a_n\}$ is **bounded above** if there exists an M such that $a_n \leq M$ for all n.

$\{a_n\}$ is **bounded below** if there exists an m such that $a_n \geq m$ for all n.

$\{a_n\}$ is **bounded** if it is bounded from above and below.

Monotonic Sequence Theorem

If $\{a_n\}$ is bounded and monotonic, then $\{a_n\}$ converges.

Practice

1. Write the first four terms of the sequence defined below. (This generates the **Fibonacci numbers**.)

   $a_1 = 1, \quad a_2 = 1, \quad a_{n+1} = a_n + a_{n-1}$

2. Find a formula for the nth terms of the sequence $5, \frac{3}{2}, \frac{1}{3}, -\frac{1}{4}, -\frac{3}{5}, -\frac{5}{6}, -1, \ldots$

3. Which of the following sequences converge, and which diverge? Find the limit of each convergent sequence.

   a) $a_n = (-1)^n \left(1 - \frac{1}{n}\right)$

   b) $a_n = 2 + (-1)^n$

   c) $a_n = \frac{2n+1}{1-3\sqrt{n}}$

d) $a_n = \left(\frac{2}{n}\right)^{1/n}$

e) $a_n = \frac{1-\cos n}{n^2}$

Q: What are the next two letters in this sequence: A E F H I K L M ?