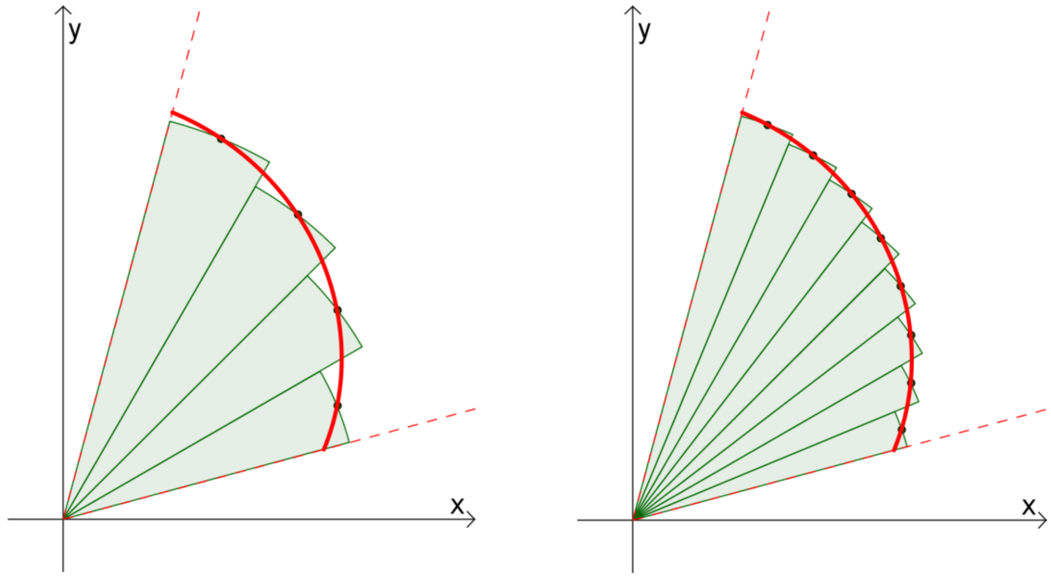
Areas and Lengths in Polar Coordinates

To find the area of a region in a polar coordinate system, we use *sectors* instead of rectangles.

(Note: the area of a sector with central angle θ is $\frac{\theta}{2\pi} \cdot \pi r^2 = \frac{1}{2}r^2\theta$.)



For a curve $r = f(\theta)$, the area of each approximating sector is: $A_k = \frac{1}{2}r_k^2\Delta\theta$

So, we can get our Riemann sums that approximate the region, and then take the limit as $n \to \infty$:

$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n} \frac{1}{2}r_k^2\Delta\theta \quad \to \quad \int_{\theta=\alpha}^{\theta=\beta} \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{r^2 d\theta} \qquad (\text{as } n \to \infty)$$

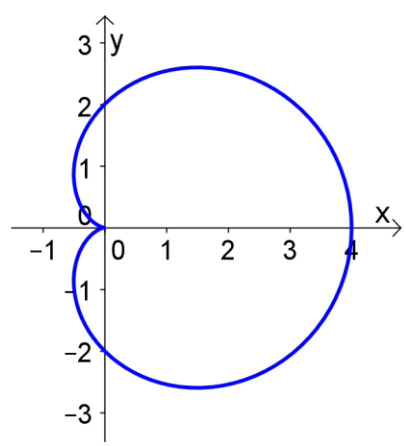
So, the **area** of the region between the origin and $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is:

$$\textbf{Area} = \int_{\theta=\alpha}^{\theta=\beta} \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{r^2 d\theta}$$

Note: As you go from $\theta = \alpha$ to $\theta = \beta$, the curve must be traversed exactly once.

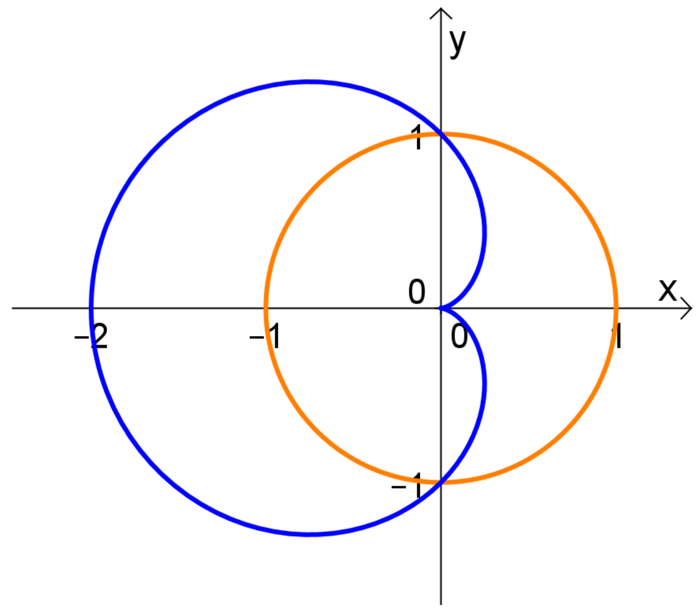
Ex 1.

Find the area of the region in the plane enclosed by the cardioid $r = 2(1 + \cos\theta)$.



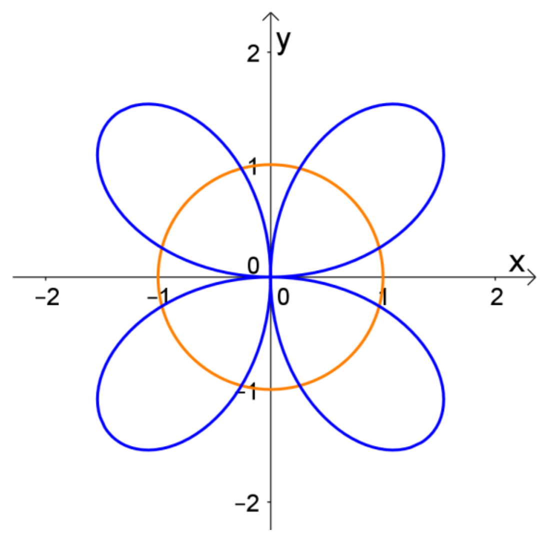
Ex 2.

Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$.



Ex 3.

Find the area of the region that lies inside both $r = 1$ and the rose $r = 2\sin 2\theta$.



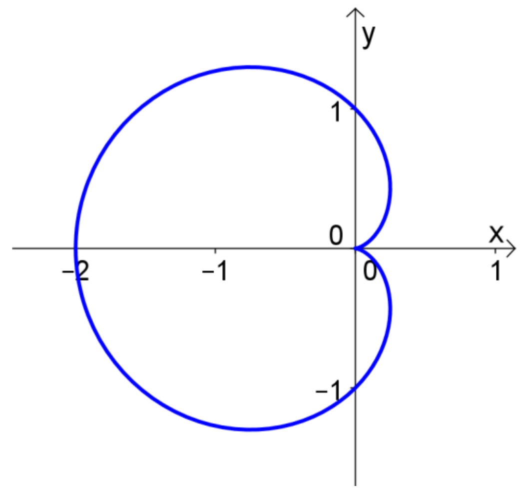
From the arc length formula for parametric equations and $x = r\cos\theta$ and $y = r\sin\theta$, we get the **arc length for a polar curve** $r = f(\theta)$ as θ goes from α to β:

$$\textbf{Arc length} = \int_{\theta=\alpha}^{\theta=\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Note: As you go from $\theta = \alpha$ to $\theta = \beta$, the curve must be traversed exactly once.

Ex 4.

Find the length of the cardioid $r = 1 - \cos\theta$.



Note: To get the arc length formula for a polar curve, we can parameterize the polar curve (in θ) and then use the arc length formula for parametric curves. Below, r is a function of θ (that is, $r(\theta)$).

$x = r\cos\theta$ and $y = r\sin\theta$

Differentiate (using the Product Rule) to get:

$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta$ and $\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\,cos\theta$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2\cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\sin^2\theta + \left(\frac{dr}{d\theta}\right)^2\sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\cos\theta + r^2\cos^2\theta$$

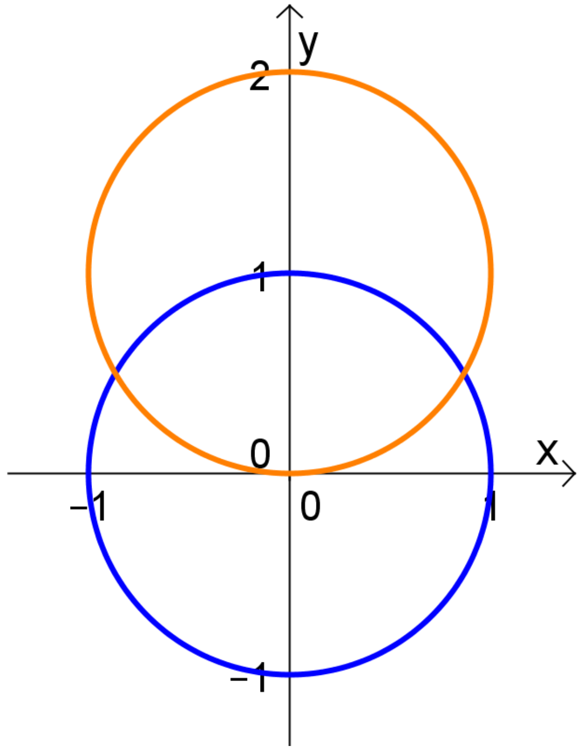
$$= \left(\frac{dr}{d\theta}\right)^2 + r^2$$

Thus,

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Practice

1. Find the area of the region shared by the circles $r = 1$ and $r = 2\sin\theta$. (see graphs to right)



2. Find the length of the curve $r = \sqrt{1+\sin 2\theta},\ 0 \le \theta \le \pi\sqrt{2}$.

Q: This is an unusual paragraph. I'm curious how quickly you can find out what is so unusual about it. It looks so plain you would think nothing was wrong with it. In fact, nothing is wrong with it! It is unusual though. Study it, and think about it, but you still may not find anything odd. But if you work at it a bit, you might find out. Try to do so without any coaching!