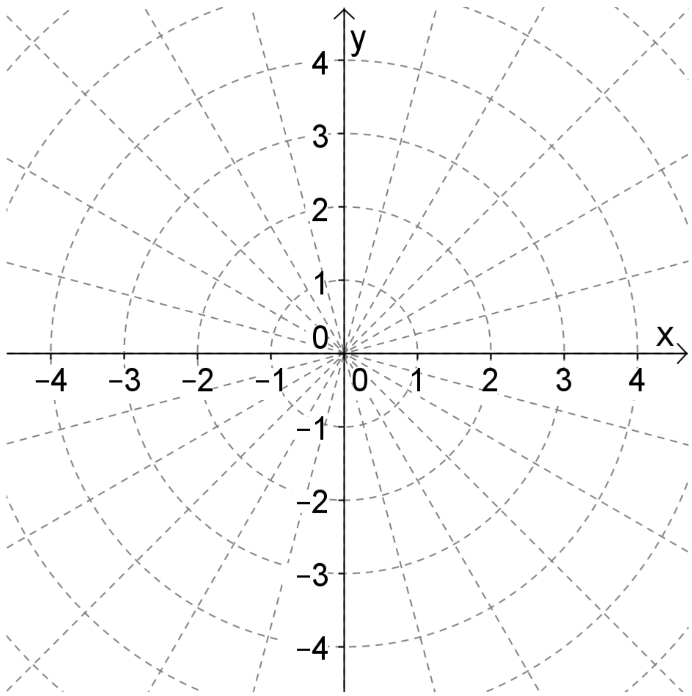
# Polar Coordinates

Recall from trigonometry that a polar coordinate system is another way to locate points on plane. So, instead of Cartesian coordinates $(x, y)$, we have polar coordinates $(r, \theta)$.




**Notes:**

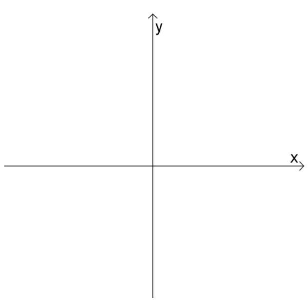
$r$ can be negative (ex: $(1,0) = (-1, \pi)$).

Each $(x, y)$ point has an infinite number of polar coordinate representations (ex: $(1,0) = (1, 2\pi k)$).

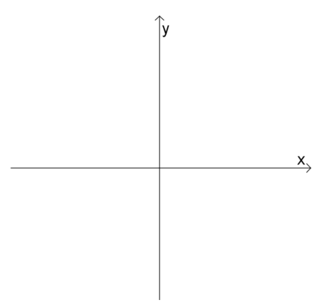
**Ex 1.**

Graph the following polar equations.

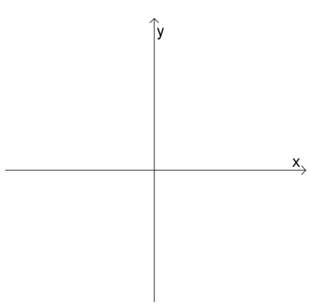
$r = 1$



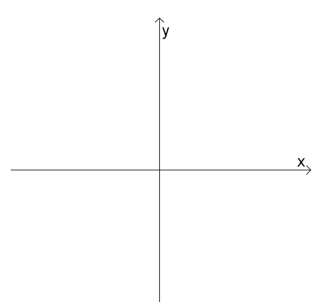

$r = -2$




$\theta = \frac{\pi}{6}$



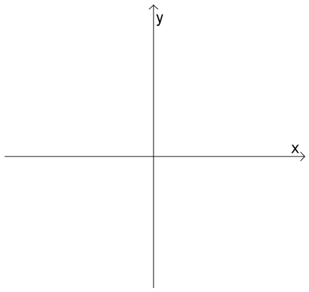

$\theta = \frac{7\pi}{6}$



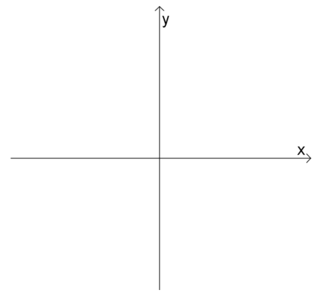
**Ex 2.**

Graph the following regions.

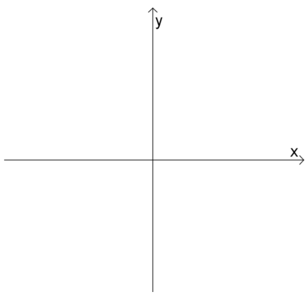
$1 \leq r \leq 3$



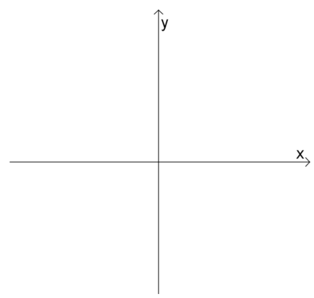

$1 \leq r \leq 3$ and $0 \leq \theta \leq \frac{\pi}{2}$




$-3 \leq r < 2$ and $\theta = \frac{\pi}{4}$




$\frac{2\pi}{3} \leq \theta \leq \frac{5\pi}{6}$




Here are the equations to get you from and to polar coordinates:

$x = r\cos\theta\,,\;\; y = r\sin\theta$ $\qquad$ $r^2 = x^2 + y^2,\;\; \tan\theta = \frac{y}{x}$

**Ex 3.**

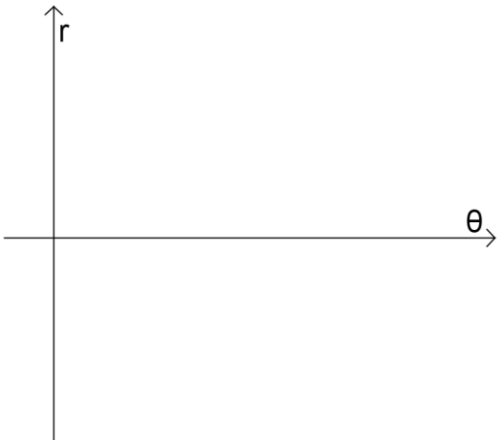
Find a polar equation for $x^2 + (y-3)^2 = 9$.

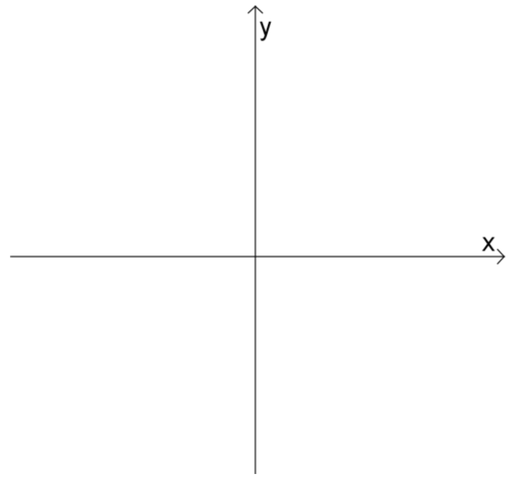
**Ex 4.**

Replace $r = \frac{4}{2\cos\theta - \sin\theta}$ with an equivalent Cartesian equation.

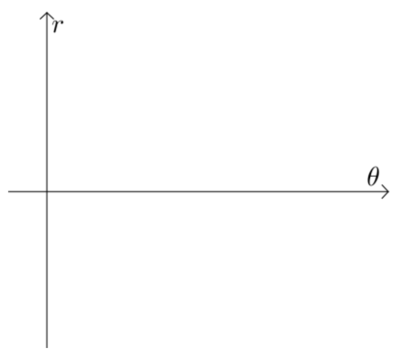
Making a sketch in the Cartesian $r\theta$-plane can help graph polar curves when you can't find an equivalent Cartesian equation. That's the main technique we'll use here.

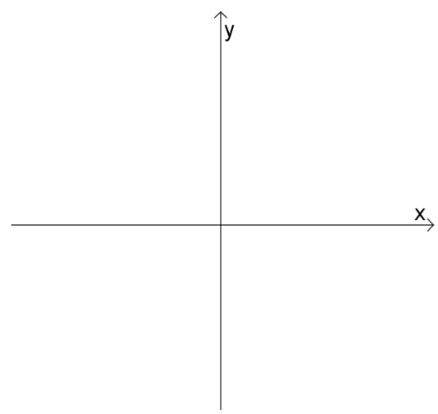
**Ex 5.**
Graph the curve $r = 1 - \cos\theta$.




**Ex 6.**
Graph the curve $r = 2 + 3\sin\theta$.




## Symmetries

Noticing symmetries can also help when graphing in polar coordinates.

If you can get back the original equation after…
…replacing $\theta$ by $-\boldsymbol{\theta}$, then graph is symmetric about $\boldsymbol{x}$**-axis**.
…replacing $\theta$ by $\boldsymbol{\pi - \theta}$, then graph is symmetric about $\boldsymbol{y}$**-axis**.
…replacing $r$ by $-\boldsymbol{r}$, then graph is symmetric about **origin**.

Some useful trig identities:
$\sin(-\theta) = -\sin\theta$
$\boldsymbol{\cos(-\theta) = \cos\theta}$
$\boldsymbol{\sin(\pi - \theta) = \sin\theta}$
$\cos(\pi - \theta) = -\cos\theta$

In Ex 5 ($r = 1 - \cos\theta$), we can see the symmetry about the $x$-axis by replacing $\theta$ with $-\theta$:
$r = 1 - \cos(-\theta)$
$r = 1 - \cos\theta$

## Slope

To find the slope of a polar equation $r = f(\theta)$, we can use the parametric equations:

$x = r\cos\theta$ and $y = r\sin\theta$ (the parameter here is $\theta$)

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

For example, what is the slope of $r = 1 - \cos\theta$ (from Ex 5) at $\theta = \frac{\pi}{2}$?

Well, since $\frac{dr}{d\theta} = \sin\theta$, we have $\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{\sin\theta\cdot\sin\theta + (1-\cos\theta)\cos\theta}{\sin\theta\cdot\cos\theta - (1-\cos\theta)\sin\theta} = \frac{\sin^2\theta + \cos\theta - \cos^2\theta}{2\sin\theta\cos\theta - \sin\theta}$.

At $\theta = \frac{\pi}{2}$, $\frac{dy}{dx} = \frac{\sin^2\frac{\pi}{2} + \cos\frac{\pi}{2} - \cos^2\frac{\pi}{2}}{2\sin\frac{\pi}{2}\cos\frac{\pi}{2} - \sin\frac{\pi}{2}} = \frac{1}{-1} = \boxed{-1}$

**Notes:**

Horizontal tangent lines will happen when $\frac{dy}{d\theta} = 0$ and $\frac{dx}{d\theta} \neq 0$.

Vertical tangent lines will happen when $\frac{dx}{d\theta} = 0$ and $\frac{dy}{d\theta} \neq 0$.

If both $\frac{dx}{d\theta} = 0$ and $\frac{dy}{d\theta} = 0$ at $\theta = \theta_0$, then you'll have to check $\lim_{\theta\to\theta_0} \frac{dy}{dx}$ and possibly use L'Hospital.

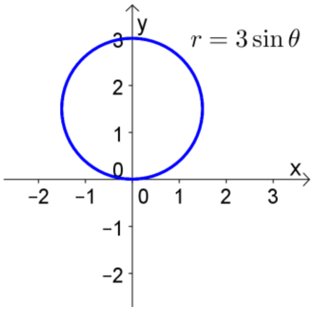
**Ex 7.**

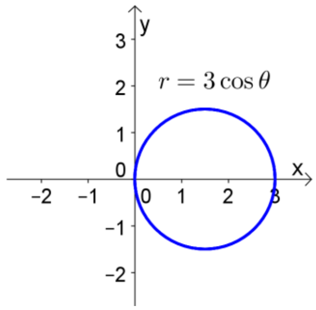
Find the values of $\theta$ in $[0, 2\pi)$ where the tangent line of $r = 1 - \cos\ \theta$ (Ex 5) is horizontal or vertical.

## Common Polar Equations and Graphs

**Circle**

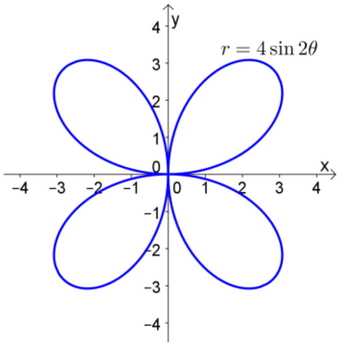
$r = a\sin\theta$ and $r = a\cos\theta$

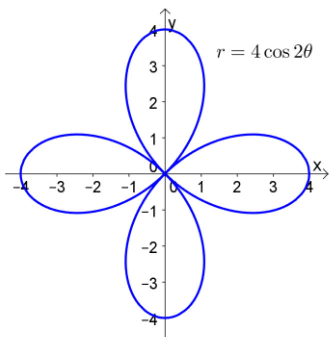


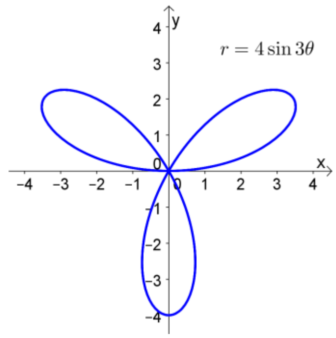




**Rose**

$r = a\sin n\theta$ and $r = a\cos n\theta$



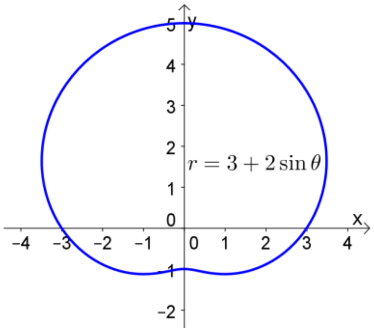



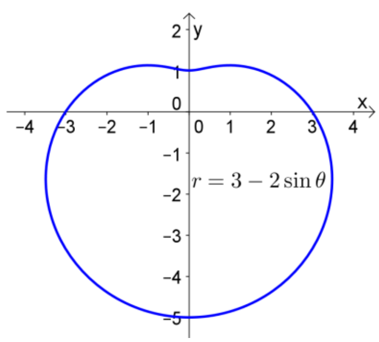




**Limacon** (pronounced “LEE-ma-sahn”)

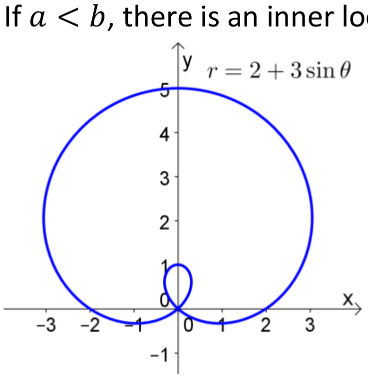
$r = a \pm b\sin\theta$ and $r = a \pm b\cos\theta$

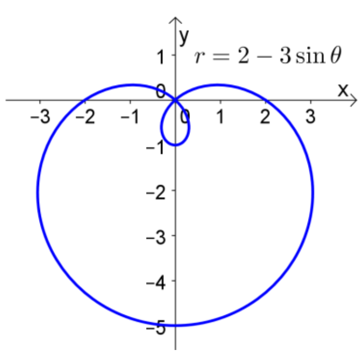
If $a > b$, there is no inner loop.



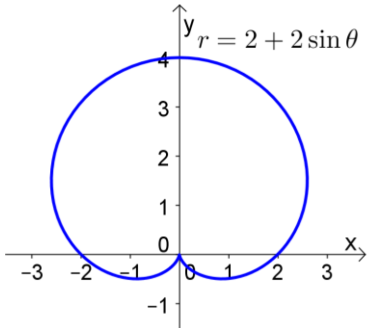



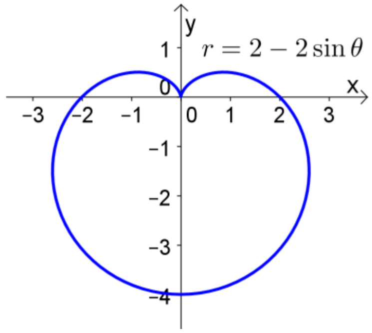

If $a < b$, there is an inner loop.



If $a = b$, it's called a cardioid (heart-shaped).

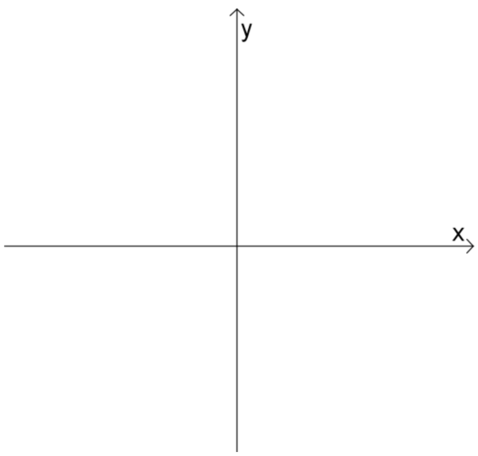


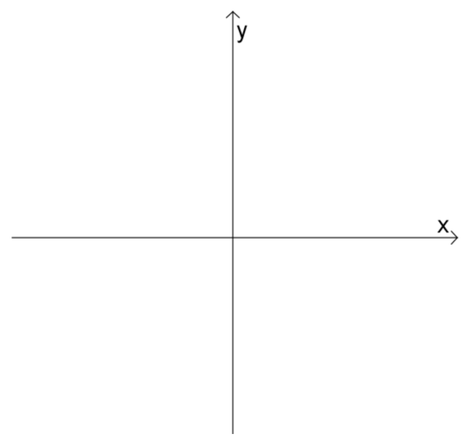

## Practice

1. Graph the following regions.

$\frac{4\pi}{3} \le \theta \le \frac{5\pi}{3}$ and $r \ge 1$

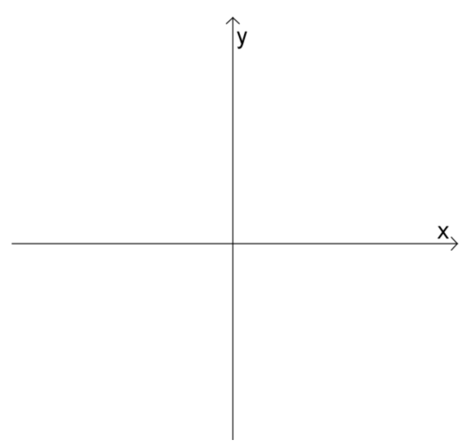



$0 \le \theta \le \frac{3\pi}{2}$ and $r = -4$

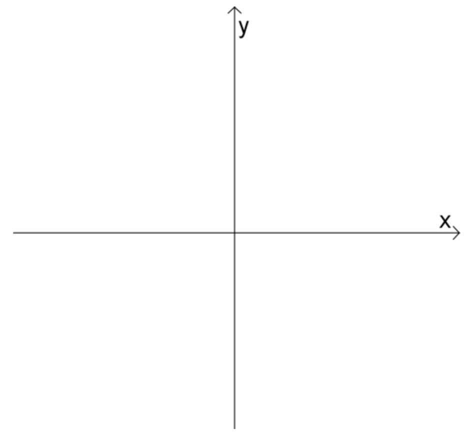



2. Find a polar equation (that has been solved for $r$) for $x^2 + y^2 = 9$.

3. Replace $r^2 = 4r\cos\theta$ with an equivalent Cartesian equation.
   Graph it.



4. Replace $r = -3\csc\theta$ with an equivalent Cartesian equation.
   Graph it.




5. Graph the curve $r = 1 + \sin\theta$.

6. Find the slope of the curve $r = \cos 2\theta$ at $\theta = 0, \frac{\pi}{2}, -\frac{\pi}{2}$, and $\pi$. Then graph the curve.

Q: Suppose your boyfriend/girlfriend sends you this text message:

A,B,C,D,E,F,G,H,I,J,K,L,M,N,O,P,Q,R,S,T,V,W,X,Y,Z.

What does the message mean?