

The Chain Rule

(covers parts of Stewart 3.4, 3.6, 3.11)

ex: If a car uses 0.2 gallons per mile when travelling 60 miles per hour, how many gallons per hour are being used?

$$(0.2 \text{ gal/mile})(60 \text{ miles/hour}) = \boxed{12 \text{ gal/hour}}$$

Ex 1.

Differentiate $y = (2x - 5)^2$.

$$\begin{array}{l} \swarrow \text{Decompose} \\ y = u^2 \quad u = 2x - 5 \\ \frac{dy}{du} = 2u \quad \frac{du}{dx} = 2 \end{array}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (2u) \cdot (2) = 4u = 4(2x - 5) = \boxed{8x - 20}$$

The Chain Rule

If $y = f(u)$ is a differentiable function of u ,

and $u = g(x)$ is a differentiable function of x ,

then $y = f(g(x))$ is a differentiable function of x , and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Here's another way to write the Chain Rule using functional notation:

$$(f \circ g)'(x) = \underbrace{f'(g(x))}_{\text{derivative of outer}} \cdot \underbrace{g'(x)}_{\text{derivative of inner}}$$

Here, think: “ f is the outer function, and g is the inner function”

Let's get some practice identifying inner and outer functions:

| | Inner function | Outer function | |
|-----------------------------|----------------------|---------------------|--------------------|
| $(x^2 + 1)^{10}$ | $x^2 + 1$ | \square^{10} | ← OR u^{10} |
| $\frac{1}{e^x + \sin x}$ | $e^x + \sin x$ | $\frac{1}{\square}$ | ← OR $\frac{1}{u}$ |
| $\sqrt{2x^2 + \frac{1}{x}}$ | $2x^2 + \frac{1}{x}$ | $\sqrt{\square}$ | ← OR \sqrt{u} |
| $\sin(\log x)$ | $\log x$ | $\sin \square$ | ← OR $\sin u$ |
| e^{x^3} | x^3 | e^{\square} | ← OR e^u |
| $\ln(5x + 2)$ | $5x + 2$ | $\ln \square$ | ← OR $\ln u$ |

Product Rule
 ↓
 $\sin x (\log x)$ and
 $\sin(\log x)$ are different!
 ↑
 Chain Rule

Ex 2.

Differentiate $\sin(x^2 + e^x)$ with respect to x .

Ex 3.

Find the derivative of $y = e^{\cos x}$.

Ex 4.

Find the derivative of $f(x) = \tanh(5 - \sin 2x)$.

Ex 5.

Find the derivative of $f(x) = \sqrt{\ln(x + 2)}$.

Ex 6.

Find the derivative of $f(x) = \ln(x^3 \cos x)$.

Ex 7.

$$\frac{d}{dx} \left(\frac{1}{3x-2} \right) =$$

Ex 8.

Differentiate $f(x) = (3x + 1)^4(2x - 1)^5$ and simplify your answer by factoring.

Ex 9.

Find all points where $f(x) = \sin^2 x$ has a horizontal tangent line.

