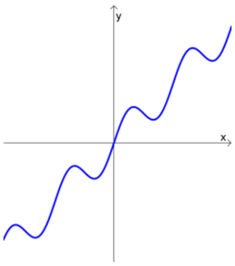
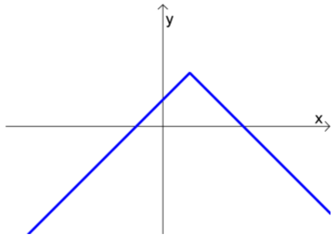
Continuity

(covers Stewart 2.5)

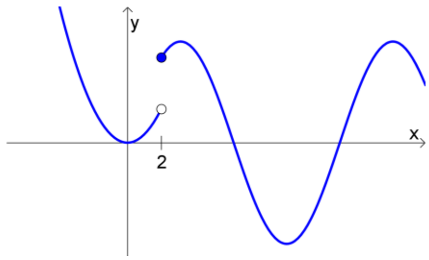
Informally, a function is continuous if you can draw it without lifting up your pencil. Here are some examples of continuous and discontinuous functions:



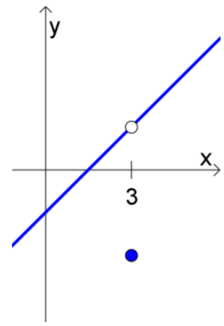
continuous



continuous



discontinuous at $x = 2$

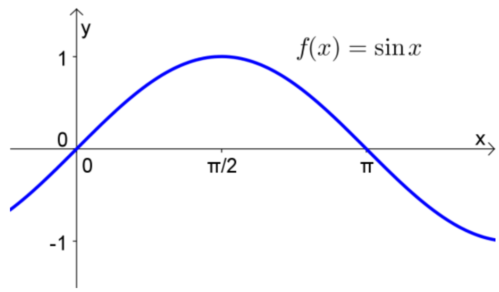


discontinuous at $x = 3$

Here's the formal definition that we'll be using:

$f(x)$ is **continuous at $x = c$** if $\boxed{\lim_{x \to c} f(x) = f(c)}$.

(Note that $f(c)$ must be defined and $\lim_{x\to c} f(x)$ must exist.)

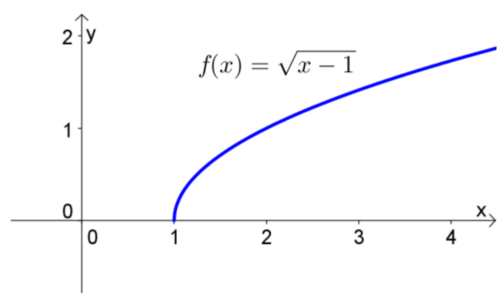


ex: $f(x) = \sin(x)$ is continuous at $x = \frac{\pi}{2}$ since $\lim_{x\to\pi/2} \sin(x) =$ ___ and $f(\pi/2) =$ ___.

We can also define left- and right-hand continuity:

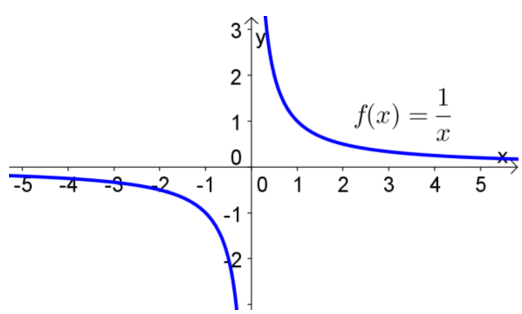
$f(x)$ is **continuous from the left at $x = c$** if $\lim_{x\to c^-} f(x) = f(c)$.

$f(x)$ is **continuous from the right at $x = c$** if $\lim_{x\to c^+} f(x) = f(c)$.



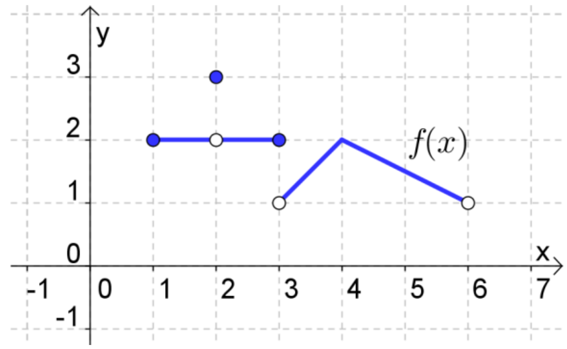
ex: $f(x) = \sqrt{x-1}$ is continuous from the right at $x = 1$ since $\lim_{x\to 1^+} \sqrt{x-1} =$ ___ and $f(1) =$ ___.

$f(x)$ is **continuous on an interval** if it is continuous at every point of the interval.



ex: $f(x) = \frac{1}{x}$ is continuous on $(0, \infty)$.

Ex 1.



Is f continuous or discontinuous at each x-value? Is f continuous or discontinuous from the left? Is f continuous or discontinuous from the right?

$\underline{x = 2}$

continuous at $x = 2$ discontinuous at $x = 2$ (circle one)

continuous from the left at $x = 2$ discontinuous from the left at $x = 2$ (circle one)

continuous from the right at $x = 2$ discontinuous from the right at $x = 2$ (circle one)

$\underline{x = 3}$

continuous at $x = 3$ discontinuous at $x = 3$ (circle one)

continuous from the left at $x = 3$ discontinuous from the left at $x = 3$ (circle one)

continuous from the right at $x = 3$ discontinuous from the right at $x = 3$ (circle one)

$\underline{x = 4}$

continuous at $x = 4$ discontinuous at $x = 4$ (circle one)

continuous from the left at $x = 4$ discontinuous from the left at $x = 4$ (circle one)

continuous from the right at $x = 4$ discontinuous from the right at $x = 4$ (circle one)

Ex 2.

Explain why $f(x) = \begin{cases} 2^{x-1} & \text{if } x < 0 \\ 3 - x & \text{if } x \geq 0 \end{cases}$ is discontinuous at $x = 0$.

Ex 3.

How would you define $f(2)$ in a way that makes $f(x) = \frac{x^2-5x+6}{x-2}$ continuous at $x = 2$? (This is called "removing the discontinuity".)

Properties of Continuous Functions

If f and g are continuous at $x = c$, then all of the following functions are also continuous at $x = c$:

$f + g$ $f - g$ $k \cdot f$ $f \cdot g$ f/g f^n $\sqrt[n]{f}$

Why? Here's the proof of $f + g$:

$$\begin{aligned} \lim_{x\to c}(f+g)(x) &= \lim_{x\to c}\big(f(x)+g(x)\big) \\ &= \lim_{x\to c} f(x) + \lim_{x\to c} g(x) \\ &= f(c) + g(c) \\ &= (f+g)(c) \end{aligned}$$

Notes:

- Any polynomial $P(x)$ is continuous since, $\lim_{x\to c} P(x) = P(c)$.
  ex: $f(x) = x^2 + 3x - 4$ is continuous everywhere (that is, $(-\infty, \infty)$).
- Any rational function $P(x)/Q(x)$ is continuous wherever $Q(x) \neq 0$.
  ex: $f(x) = \frac{x+1}{x-2}$ is continuous for all x except 2.

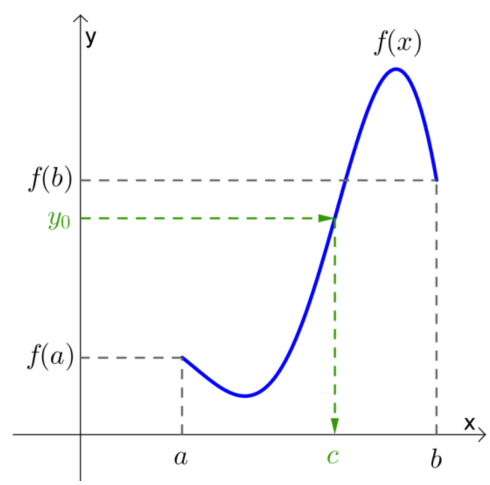
- $|x|$, $\sin x$, and $\cos x$ are continuous ________________.
- $\sqrt{x}$ is continuous ___________________.
- If f is continuous at c and g is continuous at $f(c)$, then $g \circ f$ is continuous at c.
  ex: $\sqrt{x+1}$ is continuous everywhere on its domain since $f(x) = x + 1$ is continuous and $g(x) = \sqrt{x}$ is continuous, so $(g \circ f)(x) = g(f(x)) = \sqrt{x+1}$ is continuous.

Ex 4.
At what points is the function $y = \frac{x+3}{x^2-3x-10}$ continuous?

Ex 5.
At what points is the function $y = \sec 2x$ continuous?

One useful fact about continuous functions is that they satisfy the
______________________________________.

A function has this property if it takes on all values between any two function values.



Intermediate Value Theorem
If f is a continuous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.

Ex 6.
Use the Intermediate Value Theorem to show that there is a root of $\sqrt[3]{x} = 1 - x$ in the interval $(0,1)$.