Limits (Numeric and Algebraic)

(covers parts of Stewart 2.2, 2.3, and 2.6)

We can find limits given the graph of a function. But what if we don't know the graph?
One way to estimate the limit is numerically (that is, plug in numbers).

Ex 1.

Find $\lim_{x \to 3^+} \frac{x+2}{x-3}$

x	**3.2**	**3.1**	**3.01**	**3.001**	**3.0001**	→	**3**
$x+2$	5.2	5.1	5.01	5.001	5.0001	→	
$x-3$	0.2	0.1	0.01	0.001	0.0001	→	
$\frac{x+2}{x-3}$	26	51	501	5001	50001	→	

Ex 2.

Find $\lim_{x \to 2^-} \frac{5-x}{x-2}$

x	**1.8**	**1.9**	**1.99**	**1.999**	**1.9999**	→	**2**
$5-x$	3.2	3.1	3.01	3.001	3.0001	→	
$x-2$	-0.2	-0.1	-0.01	-0.001	-0.0001	→	
$\frac{5-x}{x-2}$	-16	-31	-301	-3001	-30001	→	

Ex 3.

Find $\lim_{x \to -4^-} \frac{1-2x}{x+4}$

x	**-4.2**	**-4.1**	**-4.01**	**-4.001**	**-4.0001**	→	**-4**
$1-2x$	9.4	9.2	9.02	9.002	9.0002	→	
$x+4$	-0.2	-0.1	-0.01	-0.001	-0.0001	→	
$\frac{1-2x}{x+4}$	-47	-92	-902	-9002	-90002	→	

Making tables of numbers is cumbersome. Let's transition to doing the analysis without a table.

Ex 4.
Find $\lim_{x \to 3^-} \frac{2x+1}{x-3}$

Ex 5.
Find $\lim_{x \to -1^+} \frac{2+3x}{x+1}$

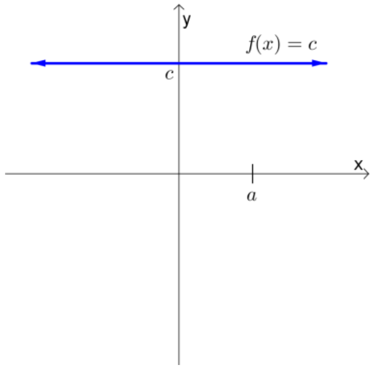
Ex 6.
Find $\lim_{x \to 2^-} \frac{2+3x}{x+1}$

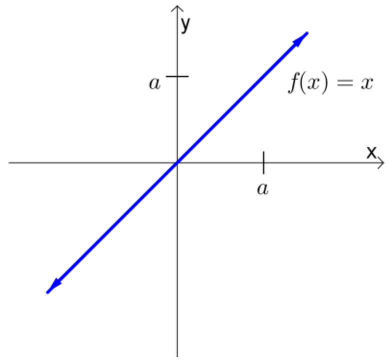
Ex 7.
Find $\lim_{x \to (\pi/2)^-} e^{\sec x}$

Ex 8.
Find $\lim_{x \to -2^+} \ln(4 - x^2)$

Ex 9.
Find $\lim_{x \to \infty} \frac{2}{\tan^{-1}(x-10)}$

Now let's get some building blocks for evaluating limits algebraically.





Note: In general, $\lim_{x \to a} c =$ ______ and $\lim_{x \to a} x =$ ________.

ex: $\lim_{x \to 71} 3 =$ ______ $\lim_{x \to 42} x =$ ______

The Limit Laws

If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ both exist, then the following laws are true:

1. $\lim_{x \to a}[f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$

2. $\lim_{x \to a}[f(x)g(x)] = \left[\lim_{x \to a} f(x)\right]\left[\lim_{x \to a} g(x)\right]$

3. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ (if $\lim_{x \to a} g(x) \neq 0$)

4. $\lim_{x \to a} cf(x) = c \lim_{x \to a} f(x)$ (here c is a constant)

5. $\lim_{x \to a}[f(x)]^p = \left[\lim_{x \to a} f(x)\right]^p$ (if $\left[\lim_{x \to a} f(x)\right]^p$ exists)

ex: Find $\lim_{x \to -2}(2x^3 - x^2 + 3)$ using the Limit Laws.

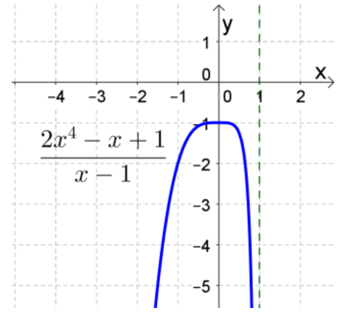
$$\begin{aligned}
\lim_{x \to -2}(2x^3 - x^2 + 3) &= \lim_{x \to -2} 2x^3 - \lim_{x \to -2} x^2 + \lim_{x \to -2} 3 \\
&= 2 \lim_{x \to -2} x^3 - \lim_{x \to -2} x^2 + \lim_{x \to -2} 3 \\
&= 2\left[\lim_{x \to -2} x\right]^3 - \left[\lim_{x \to -2} x\right]^2 + \lim_{x \to -2} 3 \\
&= 2 \cdot (-2)^3 - (-2)^2 + 3 \\
&= -16 - 4 + 3 \\
&= -17
\end{aligned}$$

Note: If $P(x)$ and $Q(x)$ are polynomials, then…

- $\lim_{x \to a} P(x) = P(a)$
- $\lim_{x \to a} \dfrac{P(x)}{Q(x)} = \dfrac{P(a)}{Q(a)}$ (if $Q(a) \neq 0$)

Ex 10.

Find $\lim_{x \to -1} \dfrac{2x^4 - x + 1}{x - 1}$



What happens when both the numerator and denominator go to 0?

Consider $f(x) = \frac{x^2-1}{x-1}$. What is $f(1)$? ________________

Now what happens to $f(x)$ as x gets close to 1?

$\boldsymbol{x}$	**0.8**	**0.99**	**0.999**	**1**	**1.001**	**1.01**	**1.2**
$\boldsymbol{f(x)}$							

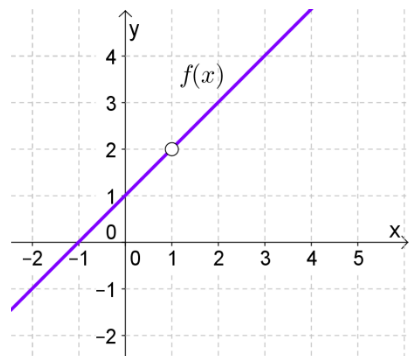
It looks like $f(x)$ approaches _____ as x approaches 1.

In other words: $\lim_{x \to 1} \frac{x^2-1}{x-1} = 2$

To the right is a picture of $f(x) = \frac{x^2-1}{x-1}$ looks like.

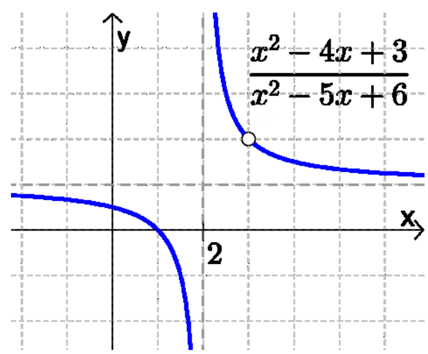
Note that we can figure the above limit out algebraically, too:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x+1)(x-1)}{x - 1} = \lim_{x \to 1}(x+1) = 1 + 1 = 2$$



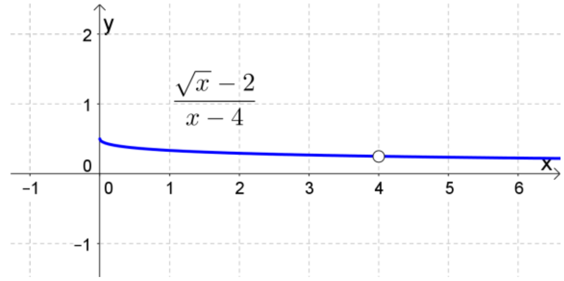
Ex 11.

$$\lim_{x \to 3} \frac{x^2 - 4x + 3}{x^2 - 5x + 6}$$



Ex 12.

Find $\lim_{x \to 4} \dfrac{\sqrt{x} - 2}{x - 4}$

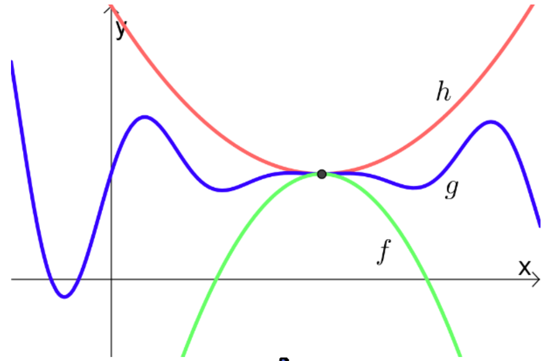


The Squeeze Theorem

Suppose that…

1. $f(x) \le g(x) \le h(x)$ for all x in some open interval containing a, except possibly at $x = a$ itself, and
2. $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$.

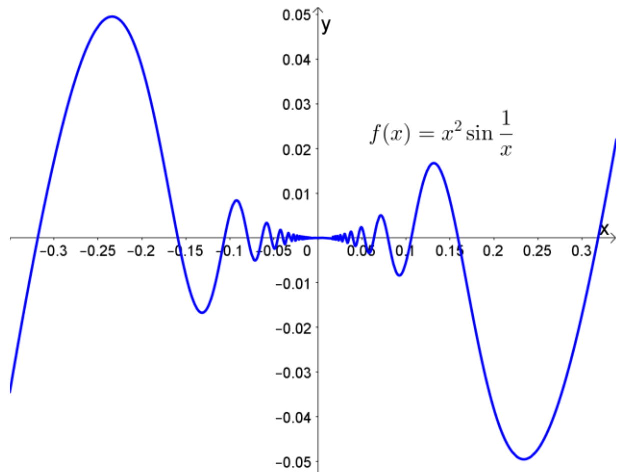
Then $\lim_{x\to a} g(x) = L$.



Note: The Squeeze Theorem has other names: Sandwich Theorem, Pinching Theorem, Two Policemen and a Drunk Theorem, etc.

Ex 13.

Use the Squeeze Theorem to prove that $\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$.



Notes:

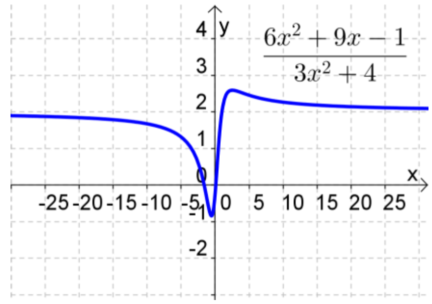
$\lim_{x\to\pm\infty} c =$ ___ (c is a constant)

$\lim_{x\to\pm\infty} \frac{1}{x} =$ ___

$\lim_{x\to\pm\infty} \frac{1}{x^2} =$ ___

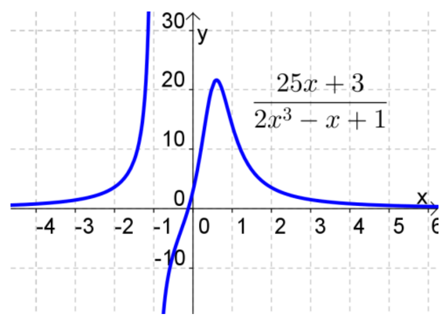
Ex 14.

$$\lim_{x\to\infty} \frac{6x^2 + 9x - 1}{3x^2 + 4}$$



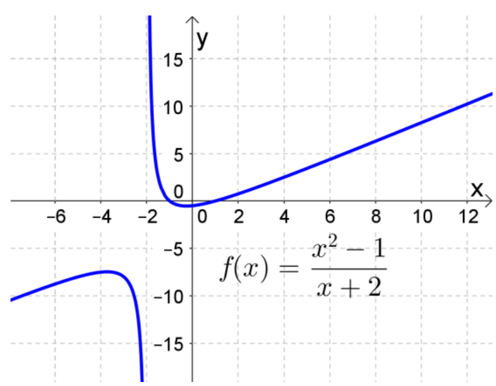
Ex 15.

$$\lim_{x\to-\infty} \frac{25x + 3}{2x^3 - x + 1}$$



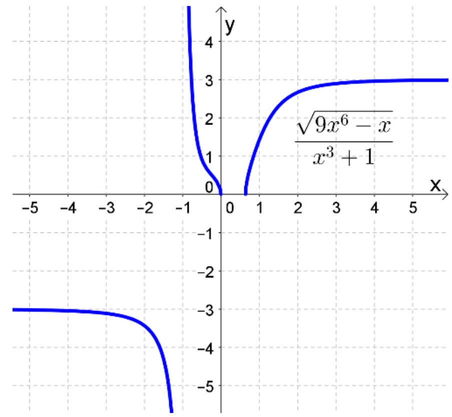
Ex 16.

$$\lim_{x\to\infty} \frac{x^2 - 1}{x + 2}$$



Ex 17.

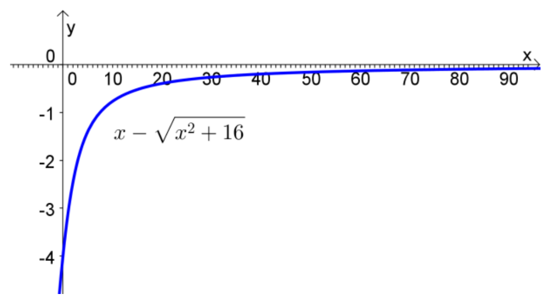
$$\lim_{x\to\infty} \frac{\sqrt{9x^6 - x}}{x^3 + 1}$$



$\infty - \infty$ Limits

Ex 18.

$$\lim_{x\to\infty} \left(x - \sqrt{x^2 + 16}\right)$$



Ex 19.

Find the horizontal asymptote of $y = \frac{\sin x}{x}$ and use the Squeeze Theorem to prove it is a horizontal asymptote.