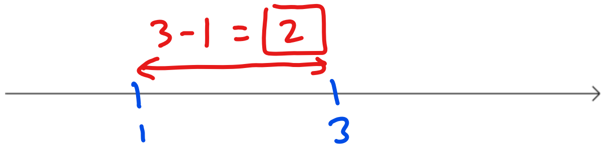
The Precise Definition of a Limit

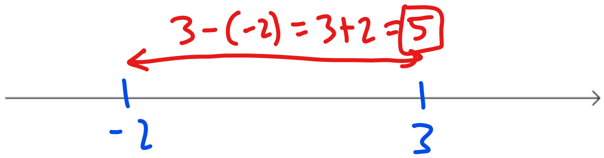
(covers Stewart 2.4)

ex: What is the distance between 1 and 3 on the number line?

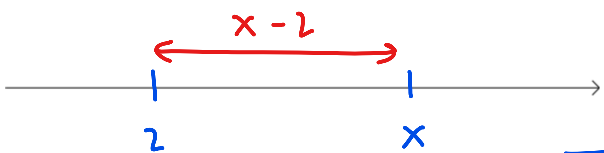


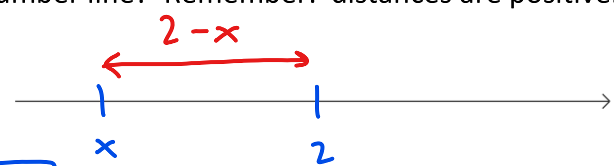
1-3 = -2

ex: What is the distance between -2 and 3 on the number line?



ex: What is the distance between x and 2 on the number line? Remember: distances are positive.





Note: We can just write $|x-2|$.

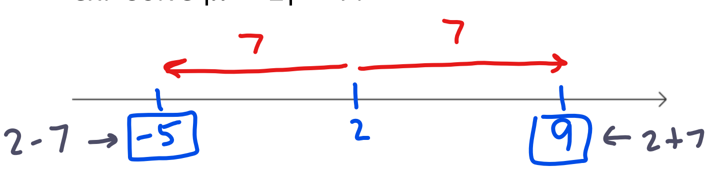
ex: What do these expressions mean on the number line?

$|x-3|$ the distance between x and 3

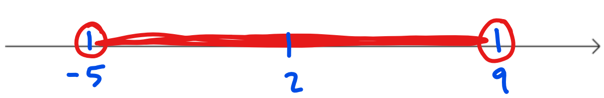
$|x+5|$ $|x-(-5)|$ the distance between x and -5

$|x|$ $|x-0|$ the distance between x and 0

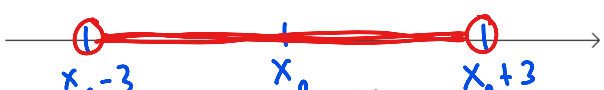
ex: Solve $|x-2| = 7$.



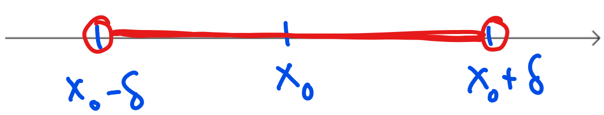
ex: Solve $|x-2| < 7$. Show the solutions on the number line.

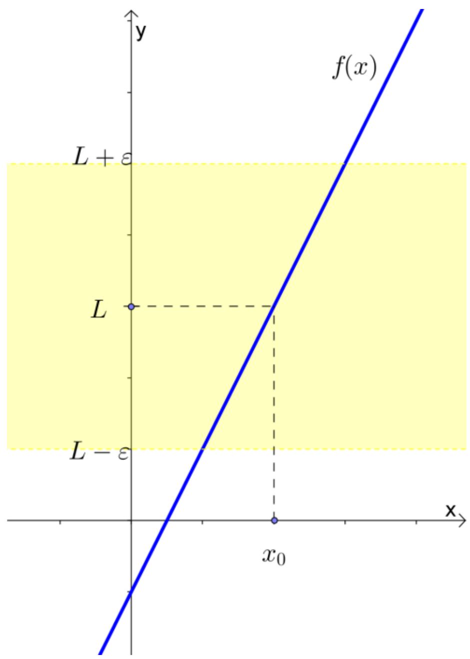


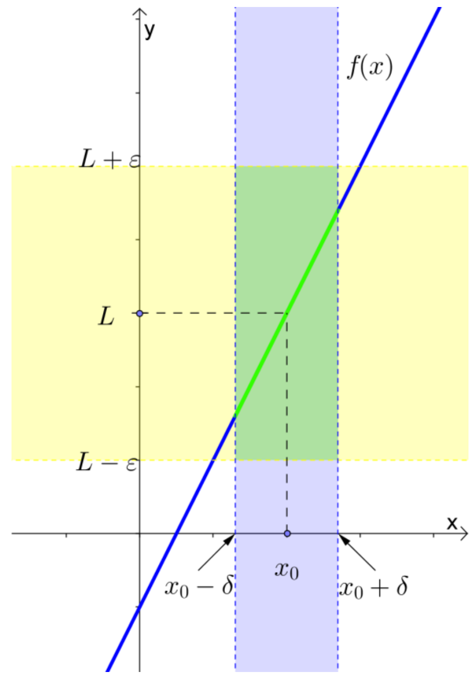
ex: Show $|x-x_0| < 3$ on the number line (x_0 is a constant).



ex: Show $|x-x_0| < \delta$ (← delta) on the number line.







Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that the **limit of $f(x)$ as x approaches x_0** is the number L, and we write

$$\lim_{x\to x_0} f(x) = L$$

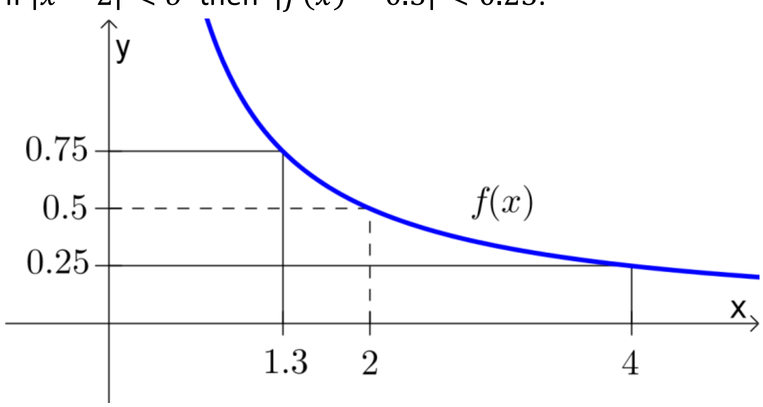
if for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon \ .$$

Note: ⇒ means "implies" (think: "if…then")

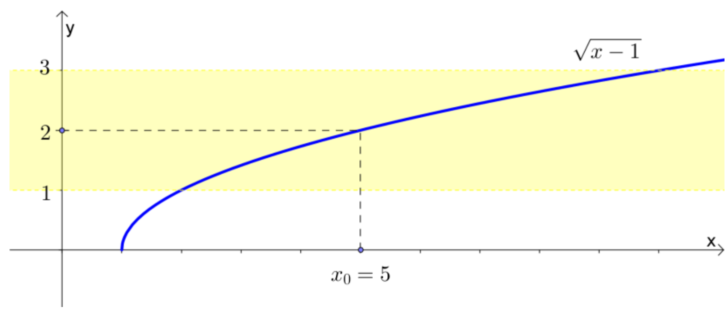
Ex 1.

Use the given graph of f to find a number δ such that if $|x - 2| < \delta$ then $|f(x) - 0.5| < 0.25$.



Ex 2.

For $\lim_{x\to 5} \sqrt{x-1} = 2$, find a $\delta > 0$ that works for $\epsilon = 1$.



Ex 3.

For $\lim_{x\to -1} 1/x = -1$, find a $\delta > 0$ that works for $\epsilon = 0.1$.

Ex 4.

Prove that $\lim_{x \to 1}(5x - 3) = 2$ by using the ϵ, δ definition of a limit.

Ex 5.

Prove that $\lim_{x \to 2} f(x) = 4$ by using the ϵ, δ definition of a limit if

$$f(x) = \begin{cases} x^2, & x \neq 2 \\ 1, & x = 2 \end{cases}$$

Note: Here is some mathematical shorthand that you might see:

∀ means "for every" or "for all"

∃ means "there exists"

s.t. means "such that".

So, the definition of the limit can be written compactly:

$$\lim_{x \to x_0} f(x) = L \text{ if } \forall \epsilon > 0, \exists \delta > 0 \text{ s.t.} \forall x, 0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon$$