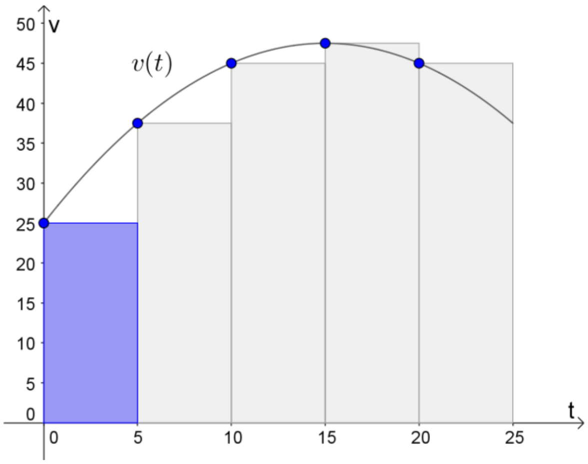
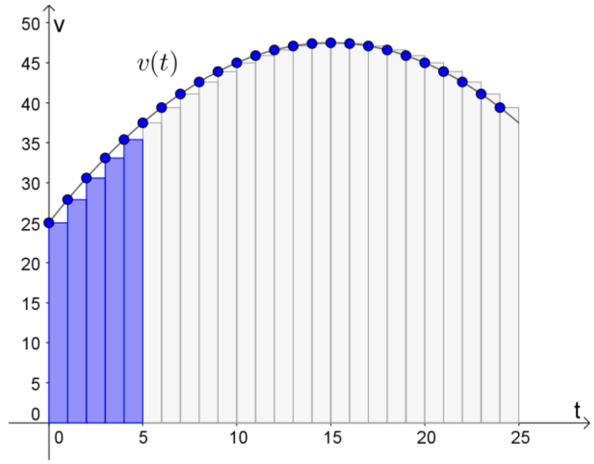
The Fundamental Theorem of Calculus

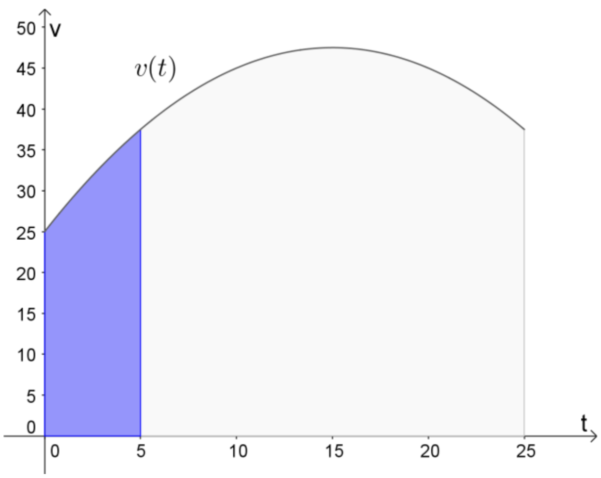
(covers Stewart 5.3 and parts of 5.4)

In packet #22, we saw how to visualize an estimation of distance traveled using velocity samples at 5-second intervals (see below, left). To get a more accurate estimation, we could sample every second (see below, right).





What if we could sample every instant?



The Fundamental Theorem of Calculus, Part 2 (also called The Evaluation Theorem)

If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Ex 1.

Evaluate.

$$\int_0^\pi \cos x\,dx$$

Notation: $[F(x)]_a^b = F(x)]_a^b = F(b) - F(a)$

Ex 2.

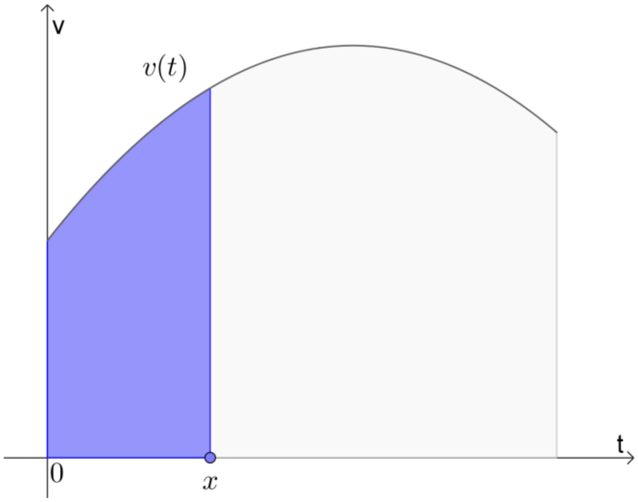
Evaluate.

$$\int_{-\frac{\pi}{4}}^0 \sec x \tan x\,dx$$

$$\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx$$

$$\int_{-3}^{-2} \frac{dx}{x+1}$$

Let's go back to the velocity function, $v(t)$. Consider the area under $v(t)$ from $t = 0$ to $t = x$ (where x is a variable here). This area represents the change in position over that time interval. Note that the area is a function of x.



The Fundamental Theorem of Calculus, Part 1

If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

Ex 3.

Use the Fundamental Theorem to find $\frac{dy}{dx}$.

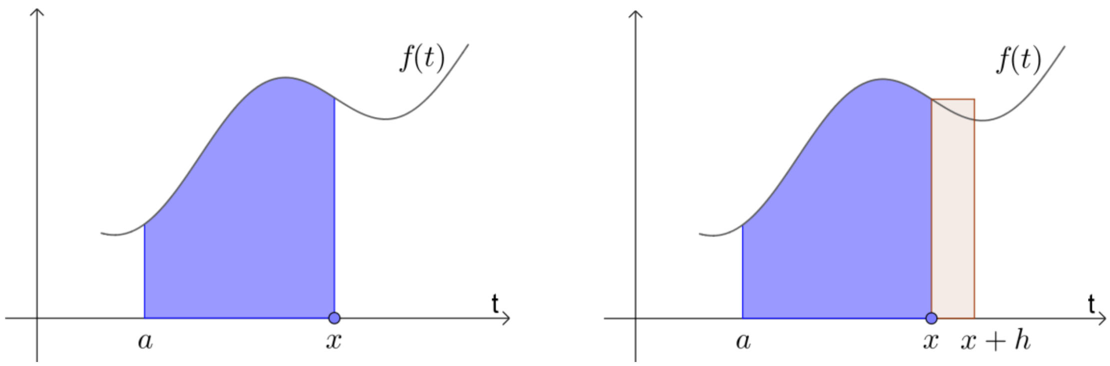
$$y = \int_a^x (t^3 + 1)\,dt$$

$$y = \int_x^5 3t \sin t\,dt$$

$$y = \int_1^{x^2} \cos t\,dt$$

$$y = \int_{1+3x^2}^{4} \frac{1}{2+e^t}\,dt$$

By the way, here's another way to get intuition about the FTOC, Part 1.



What is the rate at which the area is changing? That is, what is $F'(x)$? Let's estimate it:

$$\underbrace{F(x+h)-F(x)}_{\text{actual change in area}} \approx \underbrace{f(x)\cdot h}_{\substack{\text{estimated}\\ \text{change in area}}}$$

$$\frac{F(x+h)-F(x)}{h} \approx f(x)$$

$$\lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = f(x)$$

$$F'(x) = f(x)$$

So, $\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$.

In other words, the rate at which the area is changing is equal to the function value of f at x.

The Net Change Theorem

The net change of $F(x)$ over $[a,b]$ is:

$$F(b) - F(a) = \int_a^b F'(x)\,dx$$

ex: The net change in position (displacement) from $t = 2$ to $t = 3$ is $s(3) - s(2)$.

We could also calculate net change in position using the integral $\int_2^3 v(t)\,dt$.

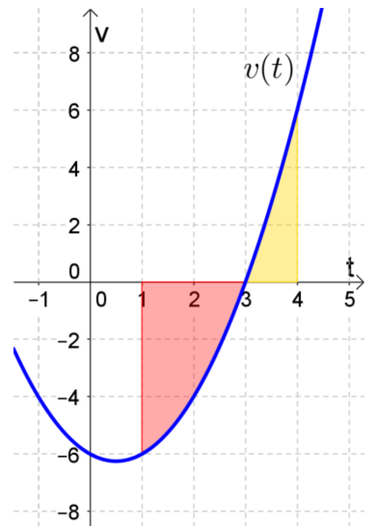
If you instead wanted total distance traveled from $t = 2$ to $t = 3$, you would calculate $\int_2^3 |v(t)|\,dt$.

ex: The net change in the volume $V(t)$ of water in a tank is $V(t_2) - V(t_1) = \int_{t_1}^{t_2} V'(t)\,dt$.

ex: The net change in population $n(t)$ is $n(t_2) - n(t_1) = \int_{t_1}^{t_2} n'(t)\,dt$.

Ex 4.

Suppose the velocity function of a particle is $v(t) = t^2 - t - 6$ (in meters per second). Find the displacement of the particle during the time period $1 \le t \le 4$.



Now find the distance traveled by the particle during the same time period $1 \le t \le 4$.