# The Definite Integral

(covers Stewart 5.2)

**Sigma notation** helps us write sums compactly. Here are some examples:

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 = \sum_{i=1}^{6} i^2$$

first term second term

i is the index

$$f(1) + f(2) + f(3) + \cdots + f(100) = \sum_{i=1}^{100} f(i)$$

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$$

$$\sum_{i=1}^{3} (-1)^i \cos i\pi = (-1)^1 \cos(1 \cdot \pi) + (-1)^2 \cos(2 \cdot \pi) + (-1)^3 \cos(3 \cdot \pi)$$
$$= 1 + 1 + 1$$
$$= 3$$

**Note:** Since we can rearrange terms, we have that

$$\sum_{i=1}^{3} (i + i^2) = (1 + 1^2) + (2 + 2^2) + (3 + 3^2)$$
$$= (1 + 2 + 3) + (1^2 + 2^2 + 3^2)$$
$$= \sum_{i=1}^{3} i + \sum_{i=1}^{3} i^2$$

In general,

$$\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n} c a_i = c \cdot \sum_{i=1}^{n} a_i \qquad \left(\text{for example, } \sum_{i=1}^{10} 2\sqrt{i} = 2 \sum_{i=1}^{10} \sqrt{i}\right)$$

$$\sum_{i=1}^{n} c = n \cdot c \qquad \left(\text{for example, } \sum_{i=1}^{99} 2 = 99 \cdot 2 = 198\right)$$

$$\underbrace{c + c + \dots + c}_{n}$$

It will be useful to know closed forms for the following summations:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

**Ex 1.**

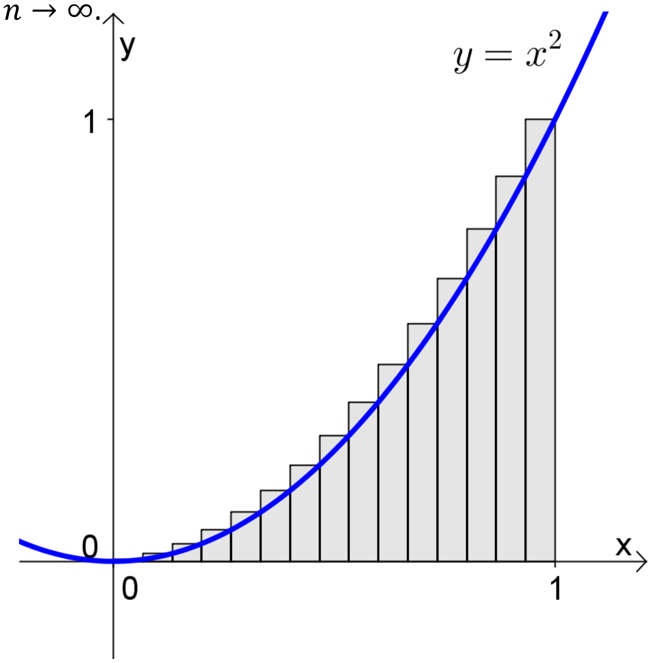
Evaluate the following sums.

$$\sum_{i=1}^{15} i^3$$

$$\sum_{i=1}^{10} i(2i+1)$$

**Ex 2.**

Write a formula for the estimation of the area under the curve $f(x) = x^2$ from $x = 0$ to $x = 1$ using $n$ rectangles and right endpoints. Then take the limit as $n \to \infty$.



## Riemann Sums

Summations like $\left(\frac{1}{n}\right)^2\left(\frac{1}{n}\right)+\left(\frac{2}{n}\right)^2\left(\frac{1}{n}\right)+\cdots+\left(\frac{n}{n}\right)^2\left(\frac{1}{n}\right)=\sum_{i=1}^{n}\left(\frac{i}{n}\right)^2\cdot\frac{1}{n}$ are called Riemann Sums.

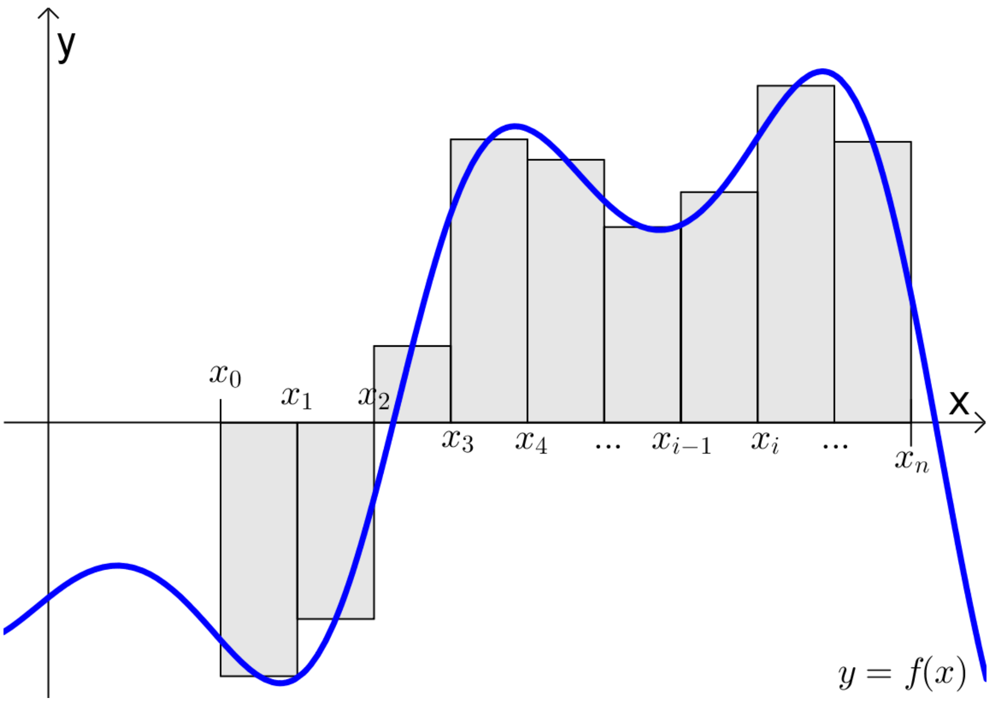
If the rectangles have the same widths, then the general Riemann Sum can be written:

$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$

Here, $x_i^*$ is any $x$-value inside each subinterval $[x_{i-1}, x_i]$. (For right endpoints, $x_i^* = a + i\Delta x$.)

So, $f(x_i^*)$ is the height of each rectangle.

$\Delta x$ is the width of each rectangle (calculated by $\frac{b-a}{n}$).




If the rectangles have different widths, then the even more general Riemann Sum is:

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

The difference here is that the rectangle widths, $\Delta x_i$, are indexed by $i$, and so can be different.

**Notes:**

1. No matter what each rectangle width is, as long as the maximum rectangle width (called the norm) approaches 0 as $n \to \infty$, then

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x_i = \text{Net area}$$

2. Rectangles below the $x$-axis count as the negative of the area in the sum. We often say that we're estimating "the area under the curve $y = f(x)$", but we're actually estimating the ***net area***, where area above the $x$-axis is counted positive, and area below the $x$-axis is counted negative.

We use the integral symbol to represent the infinite sum of infinitely thin rectangles:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

$\int_a^b f(x)\,dx$ is called a ___________________, and is read: "the integral from $a$ to $b$ of $f$ of $x$ dee $x$"

$a$ is the ______________________________.

$b$ is the ______________________________.

$f(x)$ is called the ___________________.

**Note:** $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(u)\,du$

**Theorem:** If $f$ is continuous over $[a, b]$ (with possibly a finite number of jump discontinuities), then $\int_a^b f(x)\,dx$ exists and so $f$ is said to be **integrable** over $[a, b]$.

**Properties:**

1. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

2. $\int_a^a f(x)\,dx =$ ____

3. $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

4. $\int_a^b \big(f(x) \pm g(x)\big)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5. $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

6. $(\min f)\cdot(b-a) \le \int_a^b f(x)\,dx \le (\max f)\cdot(b-a)$

7. If $f(x) \ge g(x)$ on $[a, b]$, then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$.

**Ex 3.**

Suppose that $\int_{-1}^{1} f(x)\,dx = 5$, $\int_{1}^{4} f(x)\,dx = -2$, and $\int_{-1}^{1} h(x)\,dx = 7$. Find the following.

$$\int_{4}^{1} f(x)\,dx$$

$$\int_{-1}^{1} [2f(x) + 3h(x)]\,dx$$

$$\int_{-1}^{4} f(x)\,dx$$

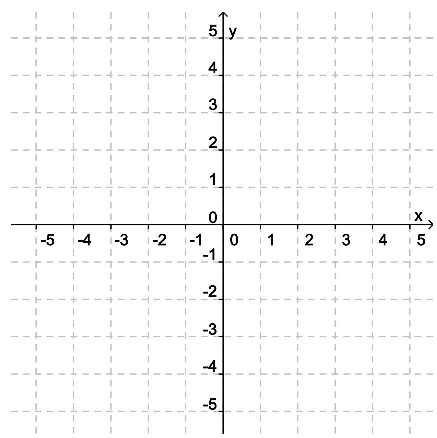
## Definite Integrals as Areas

For a nonnegative, integrable function $f(x)$, $\int_{a}^{b} f(x)\,dx$ calculates the area under the graph of $f$. If $f(x)$ is ever negative in $[a, b]$, then $\int_{a}^{b} f(x)\,dx$ computes net area.

**Ex 4.**

Graph the integrand and use the area under the graph to evaluate the integral $\int_{-4}^{0} \sqrt{16 - x^2}\,dx$.

$$\int_{-4}^{0} \sqrt{16 - x^2}\,dx$$




**Ex 5.**

Graph the integrand and use the area under the graph to evaluate the integral $\int_{-4}^{0} \left(1 + \sqrt{16 - x^2}\right)\,dx$.

$$\int_{-4}^{0} \left(1 + \sqrt{16 - x^2}\right)\,dx$$