

The Definite Integral

(covers Stewart 5.2)

Sigma notation helps us write sums compactly. Here are some examples:

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 = \sum_{i=1}^6 i^2$$

$$f(1) + f(2) + f(3) + \cdots + f(100) = \sum_{i=1}^{100} f(i)$$

$$\sum_{i=1}^n a_i = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$$

$$\begin{aligned} \sum_{i=1}^3 (-1)^i \cos i\pi &= (-1)^1 \cos(1 \cdot \pi) + (-1)^2 \cos(2 \cdot \pi) + (-1)^3 \cos(3 \cdot \pi) \\ &= 1 + 1 + 1 \\ &= 3 \end{aligned}$$

Note: Since we can rearrange terms, we have that

$$\begin{aligned} \sum_{i=1}^3 (i + i^2) &= (1 + 1^2) + (2 + 2^2) + (3 + 3^2) \\ &= (1 + 2 + 3) + (1^2 + 2^2 + 3^2) \\ &= \sum_{i=1}^3 i + \sum_{i=1}^3 i^2 \end{aligned}$$

In general,

$$\sum_{i=1}^n (a_i \pm b_i) = \sum_{i=1}^n a_i \pm \sum_{i=1}^n b_i$$

$$\sum_{i=1}^n c a_i = c \cdot \sum_{i=1}^n a_i \quad (\text{for example, } \sum_{i=1}^{10} 2\sqrt{i} = 2 \sum_{i=1}^{10} \sqrt{i})$$

$$\sum_{i=1}^n c = n \cdot c \quad (\text{for example, } \sum_{i=1}^{99} 2 = 99 \cdot 2 = 198)$$

It will be useful to know closed forms for the following summations:

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Ex 1.

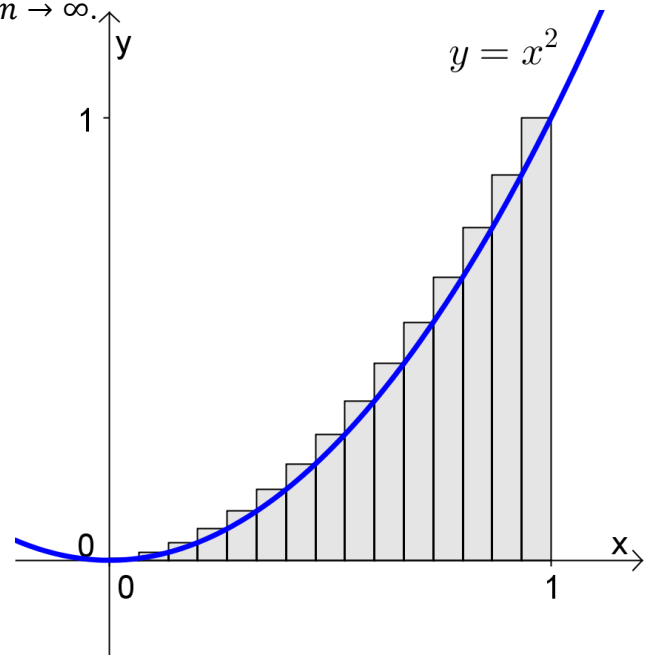
Evaluate the following sums.

$$\sum_{i=1}^{15} i^3$$

$$\sum_{i=1}^{10} i(2i+1)$$

Ex 2.

Write a formula for the estimation of the area under the curve $f(x) = x^2$ from $x = 0$ to $x = 1$ using n rectangles and right endpoints. Then take the limit as $n \rightarrow \infty$.



Riemann Sums

Summations like $\left(\frac{1}{n}\right)^2 \left(\frac{1}{n}\right) + \left(\frac{2}{n}\right)^2 \left(\frac{1}{n}\right) + \cdots + \left(\frac{n}{n}\right)^2 \left(\frac{1}{n}\right) = \sum_{i=1}^n \left(\frac{i}{n}\right)^2 \cdot \frac{1}{n}$ are called Riemann Sums.

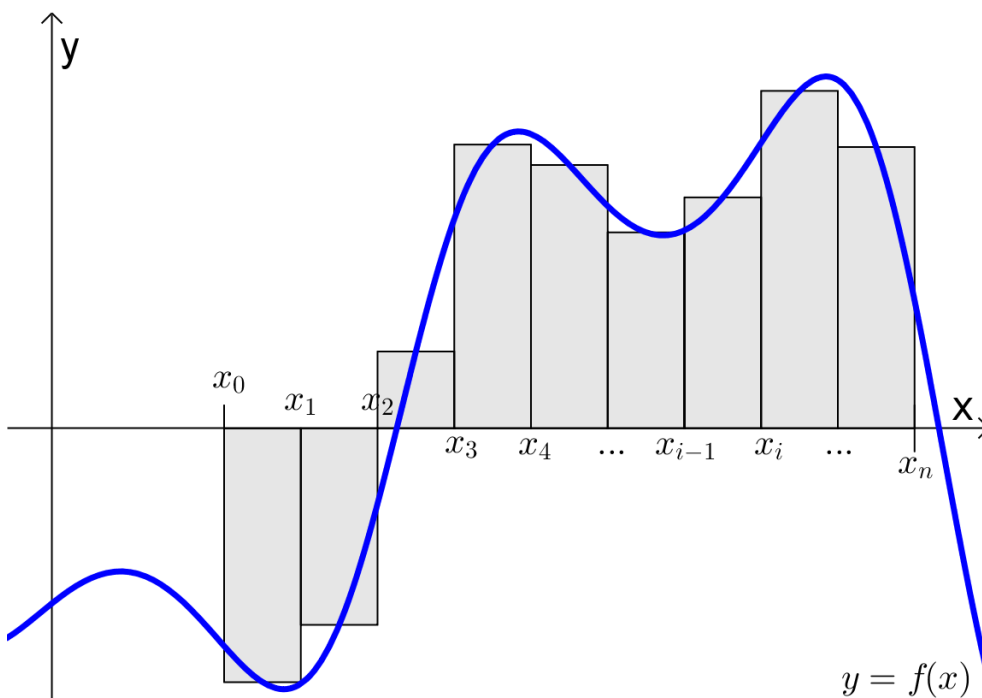
If the rectangles have the same widths, then the general Riemann Sum can be written:

$$\sum_{i=1}^n f(x_i^*) \Delta x$$

Here, x_i^* is any x -value inside each subinterval $[x_{i-1}, x_i]$. (For right endpoints, $x_i^* = a + i\Delta x$.)

So, $f(x_i^*)$ is the height of each rectangle.

Δx is the width of each rectangle (calculated by $\frac{b-a}{n}$).



If the rectangles have different widths, then the even more general Riemann Sum is:

$$\sum_{i=1}^n f(x_i^*) \Delta x_i$$

The difference here is that the rectangle widths, Δx_i , are indexed by i , and so can be different.

Notes:

1. No matter what each rectangle width is, as long as the maximum rectangle width (called the norm) approaches 0 as $n \rightarrow \infty$, then

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x_i = \text{Net area}$$

2. Rectangles below the x -axis count as the negative of the area in the sum. We often say that we're estimating "the area under the curve $y = f(x)$ ", but we're actually estimating the **net area**, where area above the x -axis is counted positive, and area below the x -axis is counted negative.

We use the integral symbol to represent the infinite sum of infinitely thin rectangles:

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$$

$\int_a^b f(x) dx$ is called a _____, and is read: "the integral from a to b of f of x dee x "

a is the _____.

b is the _____.

$f(x)$ is called the _____.

Note: $\int_a^b f(x) dx = \int_a^b f(t) dt = \int_a^b f(u) du$

Theorem: If f is continuous over $[a, b]$ (with possibly a finite number of jump discontinuities), then $\int_a^b f(x) dx$ exists and so f is said to be **integrable** over $[a, b]$.

Properties:

1. $\int_b^a f(x) dx = -\int_a^b f(x) dx$

2. $\int_a^a f(x) dx = \underline{\hspace{2cm}}$

3. $\int_a^b kf(x) dx = k \int_a^b f(x) dx$

4. $\int_a^b (f(x) \pm g(x)) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$

5. $\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$

6. $(\min f) \cdot (b - a) \leq \int_a^b f(x) dx \leq (\max f) \cdot (b - a)$

7. If $f(x) \geq g(x)$ on $[a, b]$, then $\int_a^b f(x) dx \geq \int_a^b g(x) dx$.

Ex 3.

Suppose that $\int_{-1}^1 f(x) dx = 5$, $\int_1^4 f(x) dx = -2$, and $\int_{-1}^1 h(x) dx = 7$. Find the following.

$$\int_4^1 f(x) dx$$

$$\int_{-1}^1 [2f(x) + 3h(x)] dx$$

$$\int_{-1}^4 f(x) dx$$

Definite Integrals as Areas

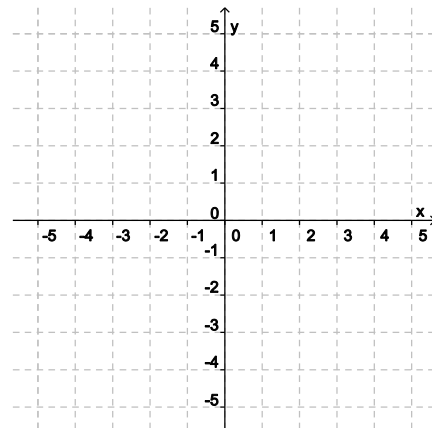
For a nonnegative, integrable function $f(x)$, $\int_a^b f(x) dx$ calculates the area under the graph of f .

If $f(x)$ is ever negative in $[a, b]$, then $\int_a^b f(x) dx$ computes net area.

Ex 4.

Graph the integrand and use the area under the graph to evaluate the integral $\int_{-4}^0 \sqrt{16 - x^2} dx$.

$$\int_{-4}^0 \sqrt{16 - x^2} dx$$

**Ex 5.**

Graph the integrand and use the area under the graph to evaluate the integral $\int_{-4}^0 (1 + \sqrt{16 - x^2}) dx$.

$$\int_{-4}^0 (1 + \sqrt{16 - x^2}) dx$$

