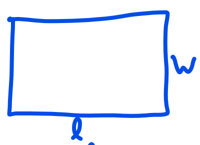
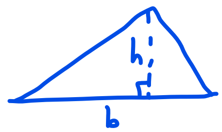
The Area Problem

(covers Stewart 5.1)

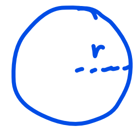
We know how to calculate areas of some shapes.



$A = lw$



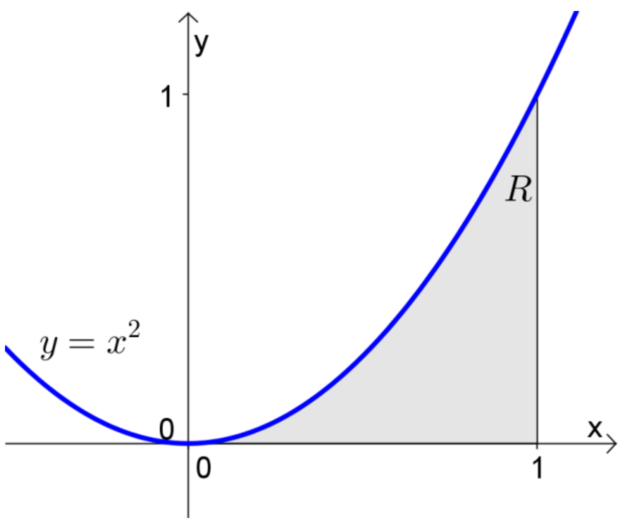
$A = \frac{1}{2}bh$

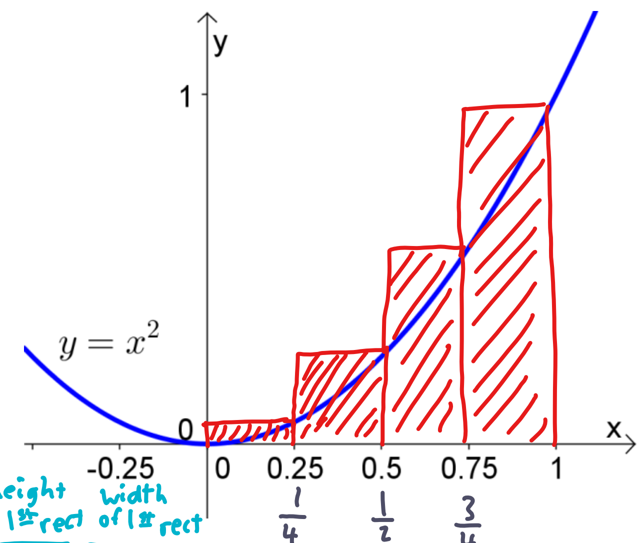


$A = \pi r^2$

Can we find a more general way to figure out areas? Let's try looking at areas under curves…

We'll start by looking at the area under the curve given by $f(x) = x^2$ from $x = 0$ to $x = 1$. Since we don't have a formula for this area, we'll use rectangles to estimate the area. For the heights of each rectangle, we'll use the maximum function value on each subinterval.





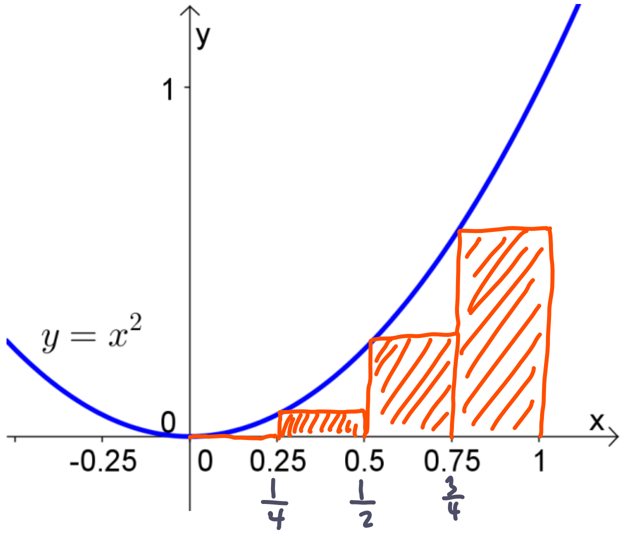
$$\text{Area of region } R = A \approx \overbrace{\left(\tfrac{1}{4}\right)^2}^{\text{height of 1st rect}} \cdot \overbrace{\tfrac{1}{4}}^{\text{width of 1st rect}} + \left(\tfrac{1}{2}\right)^2 \cdot \tfrac{1}{4} + \left(\tfrac{3}{4}\right)^2 \cdot \tfrac{1}{4} + (1)^2 \cdot \tfrac{1}{4}$$

$$= \frac{15}{32}$$

$$= 0.46875$$

When the heights of the rectangles are determined by the <u>maximum</u> function value on each subinterval, our area estimation is an <u>upper sum</u>, and it always <u>overestimates</u> the true area under the curve.

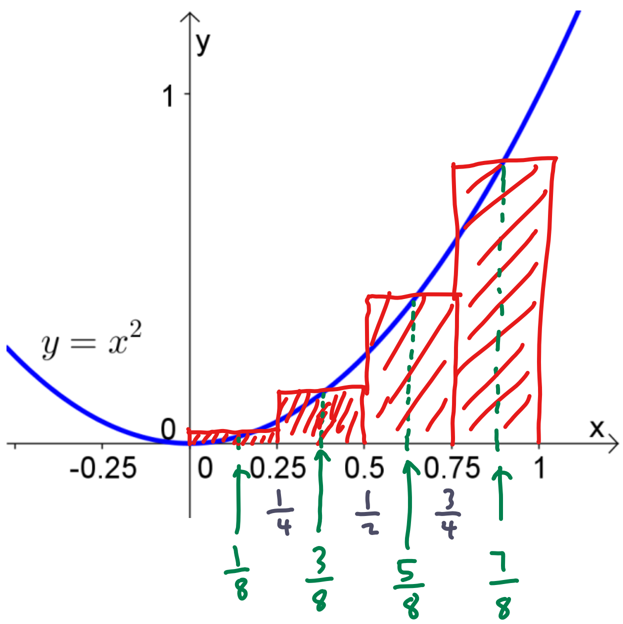
This time, let's use the minimum function value on each subinterval for the rectangle heights.



$$A \approx (0)^2 \cdot \frac{1}{4} + \left(\frac{1}{4}\right)^2 \cdot \frac{1}{4} + \left(\frac{1}{2}\right)^2 \cdot \frac{1}{4} + \left(\frac{3}{4}\right)^2 \cdot \frac{1}{4} = \frac{7}{32} = 0.21875$$

When the heights of the rectangles are determined by the __minimum__ function value on each subinterval, our area estimation is a __lower sum__, and it always __underestimates__ the true area under the curve.

We could also use the midpoint of each subinterval to get the heights of the rectangles:



$$A \approx \left(\frac{1}{8}\right)^2 \cdot \frac{1}{4} + \left(\frac{3}{8}\right)^2 \cdot \frac{1}{4} + \left(\frac{5}{8}\right)^2 \cdot \frac{1}{4} + \left(\frac{7}{8}\right)^2 \cdot \frac{1}{4}$$

$$= \frac{21}{64}$$

$$= 0.328125$$

Distance Traveled

Suppose we want to find the distance traveled during a certain time period if we know velocity.
If velocity is constant, we have $distance = velocity \times time$.
For example, if we travel at 40 mph for 2 hours, then $distance \ =$ ______________________________.

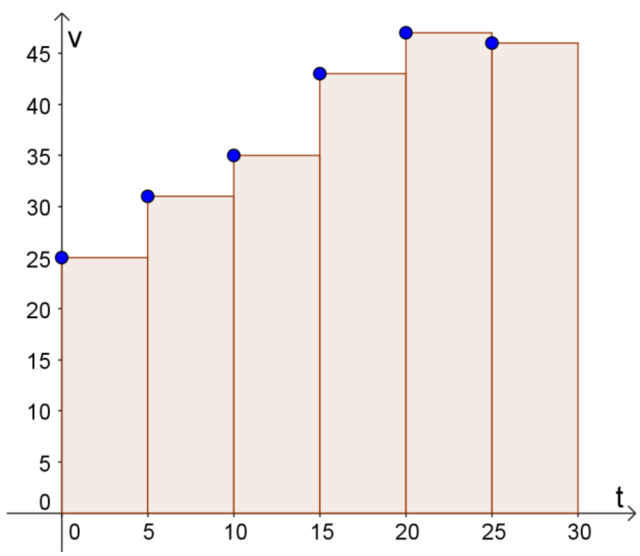
If velocity is not constant, then there's no easy formula. But we can take sample readings of velocity at a bunch of time intervals, and make estimations of the distances traveled on each time interval based on our sample velocities.

Ex 1.

Estimate the distance traveled during the following 30-second interval given the sample velocities.

Time (s)	0	5	10	15	20	25	30
Velocity (ft/s)	25	31	35	43	47	46	41

We can visualize our calculations on a coordinate system:



This gives us our first clue that the area under the curve of a function is somehow related to the antiderivative of the function…