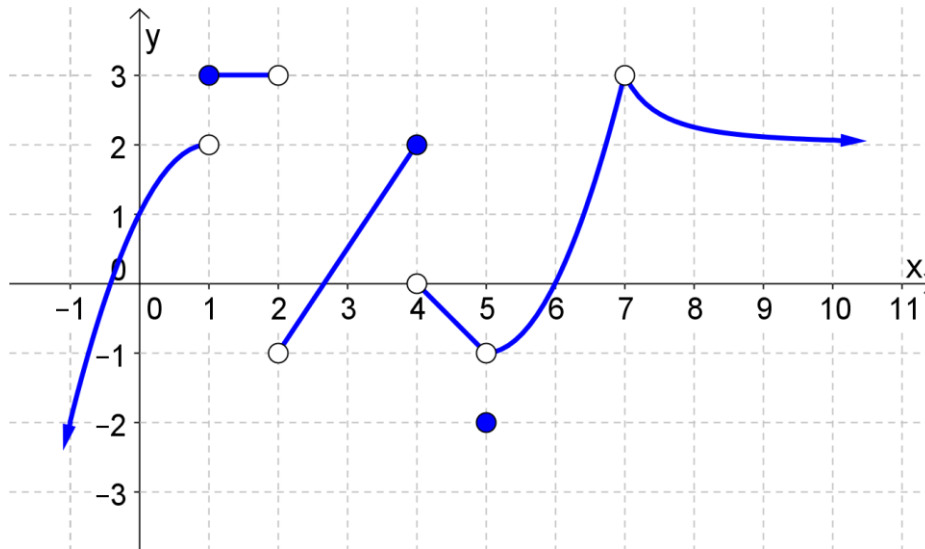


## Limits (Visual)

(covers parts of Stewart 2.2 and 2.6)

Let's warm up by visually plugging inputs into a function and reading off the outputs.



### Ex 1.

Find the following, given the graph of the crazy piecewise-defined function,  $f(x)$ , above.

$$f(0)$$

$$f(1)$$

$$f(2)$$

$$f(4)$$

$$f(5)$$

$$f(6)$$

$$f(7)$$

The **limit** of a function is a core concept in calculus. Other calculus concepts we'll explore (namely, the derivative and the integral) are defined in terms of limits. To get the intuition behind what a limit means, let's first approach limits visually. Here, you have to imagine an animation of the inputs and outputs of a function. Ask: what's happening to the outputs as the inputs are changing?

Note:  $x \rightarrow 1^-$  means "x approaches 1 from the left"

$x \rightarrow 1^+$  means "x approaches 1 from the right"

### Ex 2.

Find the following limits, given the graph of  $f(x)$  above.

$$\lim_{x \rightarrow 1^-} f(x)$$

$$\lim_{x \rightarrow 1^+} f(x)$$

$$\lim_{x \rightarrow 2^-} f(x)$$

$$\lim_{x \rightarrow 2^+} f(x)$$

$$\lim_{x \rightarrow 4^-} f(x)$$

$$\lim_{x \rightarrow 4^+} f(x)$$

$$\lim_{x \rightarrow 5^-} f(x)$$

$$\lim_{x \rightarrow 5^+} f(x)$$

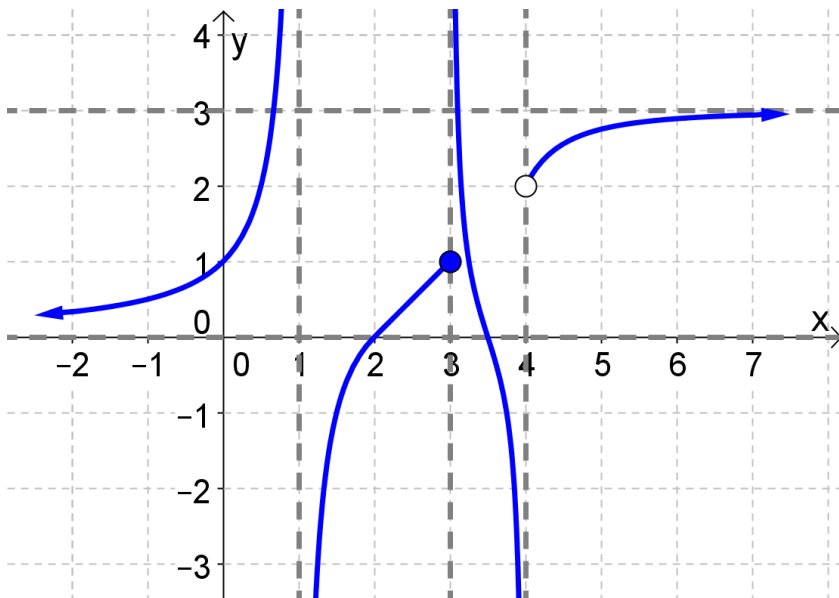
$$\lim_{x \rightarrow 6^-} f(x)$$

$$\lim_{x \rightarrow 6^+} f(x)$$

$$\lim_{x \rightarrow 7^-} f(x)$$

$$\lim_{x \rightarrow 7^+} f(x)$$

We can use  $+\infty$  and  $-\infty$  to describe the behavior of the following function.



**Ex 3.**

Find the following limits, given the graph of  $f(x)$  above.

$$\lim_{x \rightarrow 1^-} f(x)$$

$$\lim_{x \rightarrow 1^+} f(x)$$

$$\lim_{x \rightarrow 3^-} f(x)$$

$$\lim_{x \rightarrow 3^+} f(x)$$

$$\lim_{x \rightarrow 4^-} f(x)$$

$$\lim_{x \rightarrow 4^+} f(x)$$

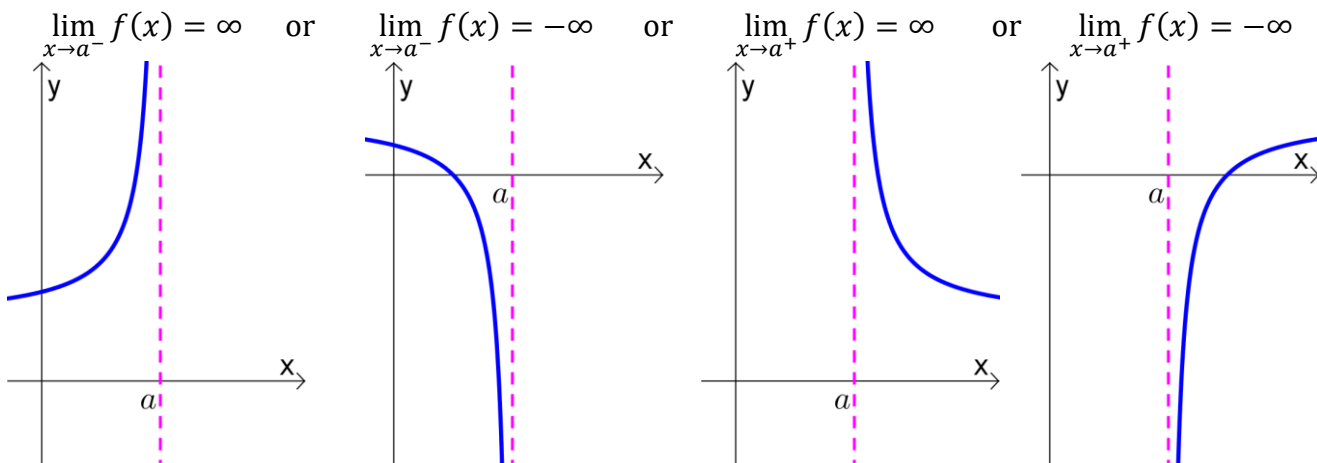
$$\lim_{x \rightarrow \infty} f(x)$$

$$\lim_{x \rightarrow -\infty} f(x)$$

Limits with  $x \rightarrow c^-$  or  $x \rightarrow c^+$  are called “one-sided” limits, because the inputs ( $x$ -values) are approaching the number  $c$  from one side (either the left or right side).

Note that if a one-sided limit is  $\infty$  or  $-\infty$ , then you have a **vertical asymptote**.

Here are the four ways that could happen:



Let's look at some familiar functions that have vertical asymptotes.

**Ex 4.**

Find the following limits.

$$\lim_{x \rightarrow \frac{\pi}{2}^-} \tan x$$

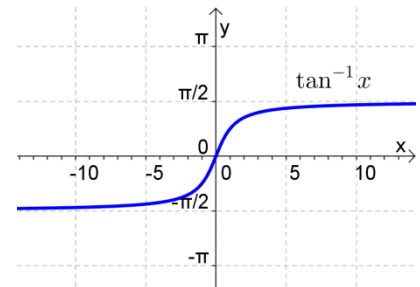
$$\lim_{x \rightarrow 0^+} \ln x$$

$$\lim_{x \rightarrow 0^+} \frac{1}{x}$$

$$\lim_{x \rightarrow 0^-} \frac{1}{x}$$

Also note that if  $\lim_{x \rightarrow \infty} f(x) = L$  or  $\lim_{x \rightarrow -\infty} f(x) = L$ , then  $f(x)$  has a **horizontal asymptote**  $y = L$ .

For example,  $\lim_{x \rightarrow \infty} \tan^{-1} x = \frac{\pi}{2}$ , so  $\tan^{-1} x$  has a horizontal asymptote  $y = \frac{\pi}{2}$ .



**Ex 5.**

Find the following limits.

$$\lim_{x \rightarrow \infty} \frac{1}{x}$$

$$\lim_{x \rightarrow -\infty} e^x$$

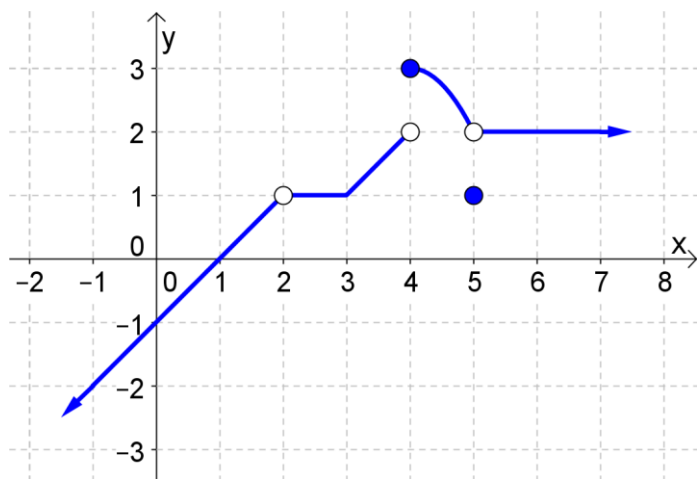
$$\lim_{x \rightarrow -\infty} 2 - e^{x+1}$$

$$\lim_{x \rightarrow \infty} 2 - e^{x+1}$$

In the future, we'll mostly use the "regular" limit, which requires the left- and right-hand limits to exist and be equal. The "regular" limit does not have a "-" or "+", and is just written using  $x \rightarrow c$ .

**Note:**  $\lim_{x \rightarrow c} f(x) = L$  if and only if  $\lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x) = L$

As with one-sided limits, regular limits only care about the behavior of the function *near*  $x = c$ , not *at*  $x = c$ .



**Ex 6.**

Find the following limits, given the graph of  $f(x)$  above.

$$\lim_{x \rightarrow 2^-} f(x)$$

$$\lim_{x \rightarrow 2^+} f(x)$$

$$\lim_{x \rightarrow 2} f(x)$$

$$\lim_{x \rightarrow 3^-} f(x)$$

$$\lim_{x \rightarrow 3^+} f(x)$$

$$\lim_{x \rightarrow 3} f(x)$$

$$\lim_{x \rightarrow 4^-} f(x)$$

$$\lim_{x \rightarrow 4^+} f(x)$$

$$\lim_{x \rightarrow 4} f(x)$$

$$\lim_{x \rightarrow 5^-} f(x)$$

$$\lim_{x \rightarrow 5^+} f(x)$$

$$\lim_{x \rightarrow 5} f(x)$$

$$\lim_{x \rightarrow 1} f(x)$$

$$\lim_{x \rightarrow 0} f(x)$$

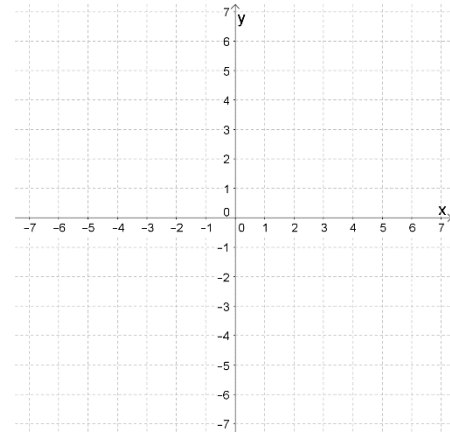
$$\lim_{x \rightarrow \infty} f(x)$$

$$\lim_{x \rightarrow -\infty} f(x)$$

Let's review how piecewise-defined functions look algebraically.

**Ex 7.**

Suppose  $f(x) = \begin{cases} x + 1 & \text{if } x < -2 \\ 4 & \text{if } -2 \leq x < 1 \\ 5 & \text{if } x = 1 \\ (x - 3)^2 & \text{if } x > 1 \end{cases}$ . Find the following limits.



$$\lim_{x \rightarrow -2^-} f(x)$$

$$\lim_{x \rightarrow -2^+} f(x)$$

$$\lim_{x \rightarrow -2} f(x)$$

$$\lim_{x \rightarrow 1^-} f(x)$$

$$\lim_{x \rightarrow 1^+} f(x)$$

$$\lim_{x \rightarrow 1} f(x)$$

In the next example, we're given information about a function, and we have to construct its graph.

**Ex 8.**

Draw the graph of a function  $f(x)$ , where  $\lim_{x \rightarrow -1^-} f(x) = \infty$ ,  $\lim_{x \rightarrow -1^+} f(x) = -\infty$ ,  $\lim_{x \rightarrow 2} f(x) = 4$ ,  $f(2) = 3$ ,  $\lim_{x \rightarrow -\infty} f(x) = 1$ , and  $\lim_{x \rightarrow \infty} f(x) = -\infty$ .

