First Derivatives and Graphs

(covers parts of Stewart 4.1 and 4.3)

<u>Recall from Precalculus:</u>

Let $f(x)$ be defined on the interval $a < x < b$. Suppose x_1 and x_2 are in that interval.
f is <u>increasing</u> on that interval if $f(x_2) > f(x_1)$ whenever $x_2 > x_1$ (so, increasing = up and right)
f is <u>decreasing</u> on that interval if $f(x_2) < f(x_1)$ whenever $x_2 > x_1$ (so, decreasing = down and right)

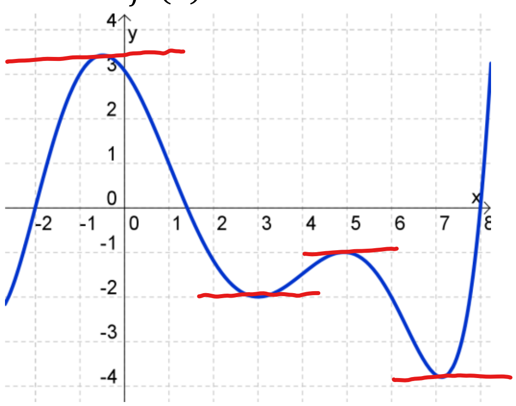
Since $f'(x)$ is the slope of the tangent line at any point on the graph of $f(x)$, $f'(x)$ tells us if $f(x)$ is increasing (positive slope) or decreasing (negative slope). In other words…

If $\boldsymbol{f'(x) > 0}$ **(positive)** on an interval, then f is **increasing** on that interval.
If $\boldsymbol{f'(x) < 0}$ **(negative)** on an interval, then f is **decreasing** on that interval.

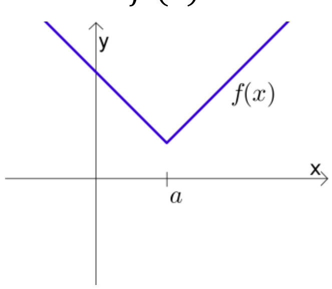
Where might $f(x)$ change from increasing to decreasing, or decreasing to increasing?
$(f'>0)$ $(f'<0)$ $(f'<0)$ $(f'>0)$

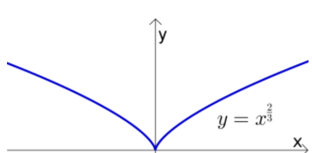
That is, where might $f'(x)$ change signs.

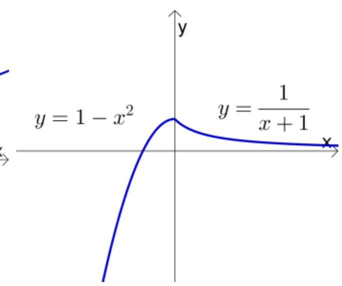
1. When $f'(x) = 0$

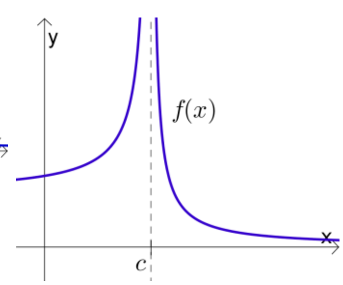


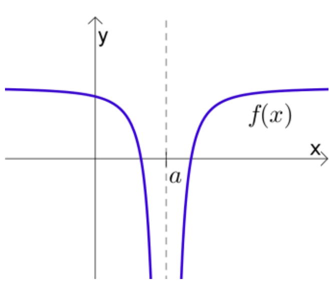
2. When $f'(x)$ DNE

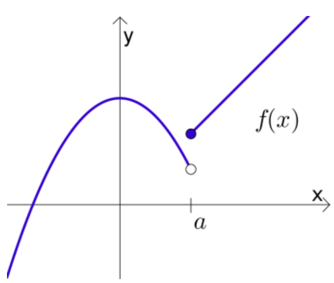


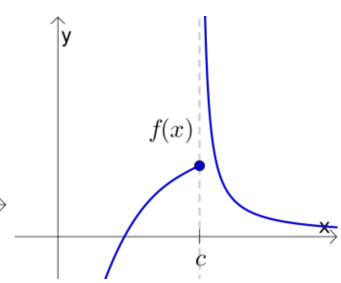






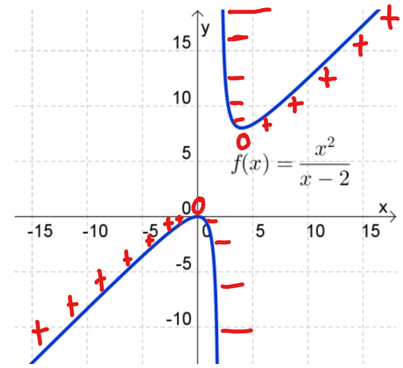






Ex 1.

Find the intervals on which $f(x) = \frac{x^2}{x-2}$ is increasing or decreasing.



Quotient Rule

$$f'(x) = \frac{(x-2)(2x) - x^2(1)}{(x-2)^2}$$

$$= \frac{2x^2 - 4x - x^2}{(x-2)^2}$$

$$= \frac{x^2 - 4x}{(x-2)^2}$$

$$= \frac{x(x-4)}{(x-2)^2}$$

$f'(x) = 0$:

$$\frac{x(x-4)}{\cancel{(x-2)^2}} = 0$$

$$x(x-4) = 0$$

$x = 0 \quad x = 4$

$f'(x)$ DNE:

$$(x-2)^2 = 0$$

$$x = 2$$

x: 0, 2, 4

$f'(x)$: $+$ 0 $-$ DNE $-$ 0 $+$

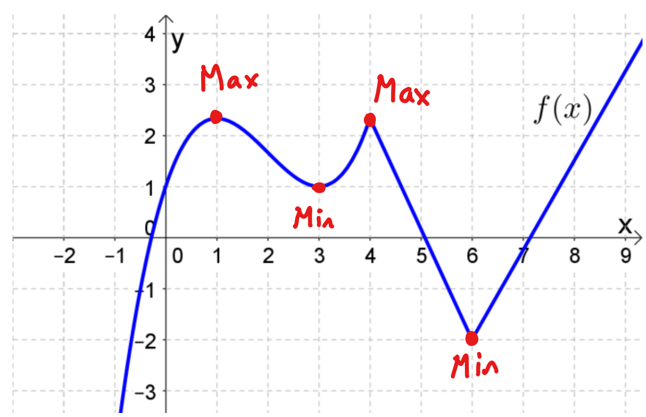
$\boxed{\text{Increasing: } (-\infty, 0) \text{ and } (4, \infty) \quad \text{Decreasing: } (0, 2) \text{ and } (2, 4)}$

$x = -1$: $f'(-1) = \frac{(-1)(-1-4)}{(-1-2)^2} = \frac{(-)(-)}{(+)}$

$x = 1$: $f'(1) = \frac{(+)(-)}{(+)}$

Local Extrema

Graphs of functions often have "peaks" and "valleys."
These are called **local extrema**.
A "peak" is called a **local maximum**.
A "valley" is called a **local minimum**.



Here are the more formal definitions for a local max and local min of a function f with domain D:

- f has a **local maximum** at $c \in D$ if $f(c) \geq f(x)$ for all $x \in D$ in some open interval containing c.
- f has a **local minimum** at $c \in D$ if $f(c) \leq f(x)$ for all $x \in D$ in some open interval containing c.

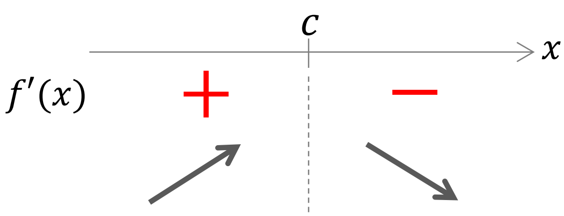
To have a local max, the function needs to be increasing and then decreasing.
To have a local min, the function needs to be decreasing and then increasing.

So, where might $f(x)$ have a local max or a local min?

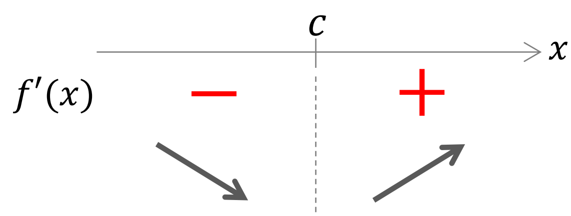
1. When $f'(x) = 0$
2. When $f'(x)$ DNE

First Derivative Test

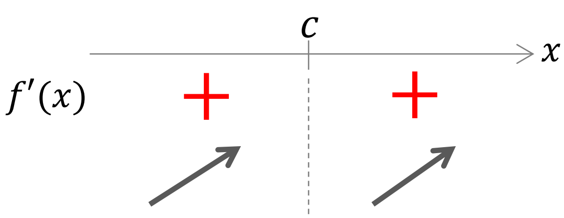
Suppose $f(c)$ is defined, and $f'(c) = 0$ or $f'(c)$ DNE.



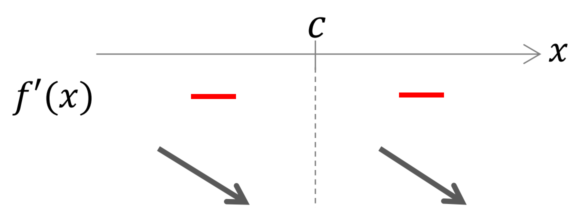
Local
max at
$(c, f(c))$



Local
min at
$(c,\ f(c))$



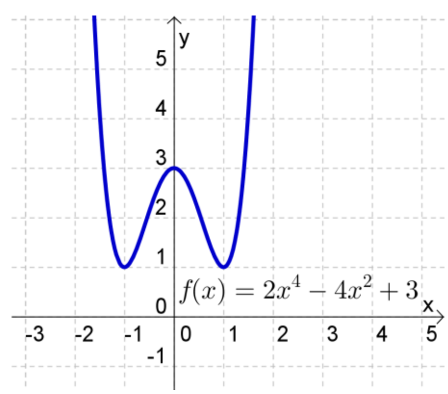
Not a local
extremum



Not a local
extremum

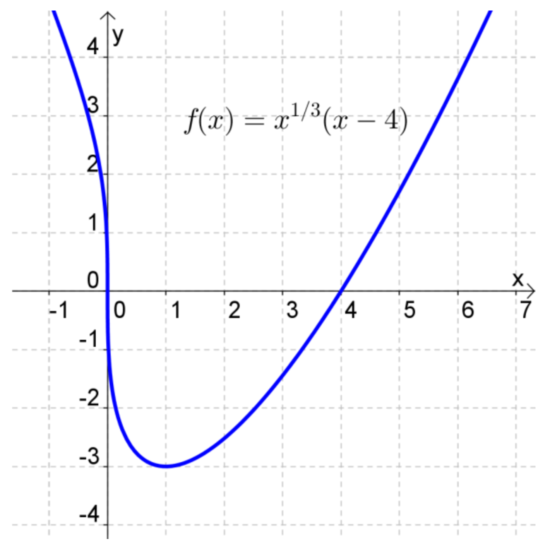
Ex 2.

Find the local maximum and minimum points of $f(x) = 2x^4 - 4x^2 + 3$.



Ex 3.

Find the intervals on which $f(x) = x^{1/3}(x-4)$ is increasing or decreasing. Also, find all points where f has a local maximum or local minimum.



Note:

f has a **critical number** at c if $f(c)$ is defined, and $f'(c) = 0$ or $f'(c)$ DNE.

The critical numbers of a function give us a list of all possible candidates for local mins/maxs.