L'Hospital's Rule

(covers Stewart 4.4)

Let's revisit limits for a moment. Consider the following limit:

$$\lim_{x\to 1}\frac{x^2-1}{x-1} = \lim_{x\to 1}\frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}} = \lim_{x\to 1}(x+1) = 2$$

As $x \to 1$, the top$\to 0$ and the bottom$\to 0$. Limits of the type $\frac{0}{0}$ are said to be indeterminate.

Previously, we would try to cancel a factor and then take the limit again. Now let's look at another powerful tool for evaluating such indeterminate limits.

L'Hôpital

L'Hospital's Rule

Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then,

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Ex 1.

Use L'Hospital's Rule to find the following limits.

$$\lim_{x\to 1}\frac{x^2-1}{x-1} \leftarrow \frac{0}{0}$$

L'Hospital's

$$= \lim_{x\to 1}\frac{2x}{1}$$

$$= 2(1) = \boxed{2}$$

$$\lim_{x\to 0}\frac{x-\sin x}{x^3} \leftarrow \frac{0}{0}$$

L'Hosp

$$= \lim_{x\to 0}\frac{1-\cos x}{3x^2} = \lim_{x\to 0}\frac{\sin x}{6x} = \lim_{x\to 0}\frac{\cos x}{6} = \frac{\cos 0}{6} = \boxed{\frac{1}{6}}$$

$\uparrow \frac{0}{0}$ L'Hosp $\uparrow \frac{0}{0}$ L'Hosp

$$\lim_{x\to 0}\frac{1-\cos x}{x+x^2} \leftarrow \frac{0}{0}$$

L'Hosp

$$= \lim_{x\to 0}\frac{\sin x}{1+2x} = \frac{\sin 0}{1+2(0)} = \frac{0}{1} = \boxed{0}$$

$\uparrow \frac{0}{1}$

Note: L'Hospital's Rule also applies to one-sided limits as well (like $\lim_{x\to 0^+}\frac{\sin x}{x^2}$).

Type $\frac{\infty}{\infty}$

L'Hospital's Rule also works for indeterminate forms of type $\frac{\infty}{\infty}$.

Ex 2.

Find the following limits.

$$\lim_{x\to\infty}\frac{x^2}{e^{3x}} \leftarrow \frac{\infty}{\infty}$$

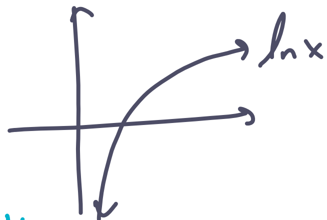
L'Hosp

$$= \lim_{x\to\infty}\frac{2x}{3e^{3x}} \leftarrow \frac{\infty}{\infty}$$

L'Hosp

$$= \lim_{x\to\infty}\frac{2}{9e^{3x}}$$

$$= \boxed{0}$$



$$\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}}$$

$$\lim_{x\to\infty}\frac{\ln x}{2\sqrt{x}} \leftarrow \frac{\infty}{\infty}$$

L'Hosp

$$= \lim_{x\to\infty}\frac{(1/x)}{\cancel{2}\left(\frac{1}{\cancel{2}x^{1/2}}\right)}$$

$$\frac{1}{x}\cdot\frac{x^{1/2}}{1} = \frac{x^{1/2}}{x} = \frac{1}{x^{1/2}}$$

$$= \lim_{x\to\infty}\frac{1}{x^{1/2}}$$

$$= \boxed{0}$$

Type $\infty \cdot 0$ and $\infty - \infty$

Sometimes we can use algebra to convert limits of type $\infty \cdot 0$ or $\infty - \infty$ into type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ (then use L'Hospital's Rule).

1	10	100	1000	10000	$\to \infty$
2	$\frac{2}{10}$	$\frac{2}{100}$	$\frac{2}{1000}$	$\frac{2}{10000}$	$\to 0$
2	2	2	2	2	$\to 2$

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = -x^{-2} = \frac{-1}{x^2}$$

Ex 3.

Find the following limits.

$$\lim_{x\to\infty}\left(x \cdot \sin\frac{1}{x}\right) \leftarrow \infty \cdot 0$$

$$= \lim_{x\to\infty}\frac{\sin\frac{1}{x}}{(1/x)} = \lim_{x\to\infty}\frac{\left(\cos\frac{1}{x}\right)\cdot\left(\cancel{-\frac{1}{x^2}}\right)}{\left(\cancel{-\frac{1}{x^2}}\right)} = \lim_{x\to\infty}\cos\frac{1}{x} = \boxed{1}$$

$\uparrow \frac{0}{0}$

L'Hosp

$$\lim_{x\to 0^+}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$$

Type $\mathbf{1}^{\infty}$, $\mathbf{0}^{\mathbf{0}}$, and $\infty^{\mathbf{0}}$

For these, try taking the logarithm of the function first, then take the limit, then exponentiate.

Ex 4.

Find the following limits.

$$\lim_{x\to 0^+}(1+x)^{1/x}$$

Derivatives of $\sin x$ and $\cos x$

As awesome as L'Hospital is, it can lead to circular reasoning. For example, what is the derivative of $\cos x$ using the definition of the derivative?

$$\frac{d}{dx}(\cos x) = \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} \underbrace{=}_{\text{L'Hospital}} \lim_{h\to 0}\frac{-\sin(x+h)}{1} = -\frac{\sin(x+0)}{1} = -\sin x$$

Why is this circular reasoning? Because to use L'Hospital, you need to know the derivative of $\cos x$. But that's what we're trying to find here! D'oh!

That's why we had to use the techniques we did when we proved the derivatives of $\sin x$ and $\cos x$.