Proving Derivatives

(covers parts of Stewart 3.1, 3.3, 3.5, 3.6)

The Power Rule

Here's the proof of the derivative of $f(x) = x^n$:

Suppose $f(x) = x^n$. Then,

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h\to 0} \frac{x^n + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + h^n - x^n}{h} \\
&= \lim_{h\to 0} \frac{nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + h^n}{h} \\
&= \lim_{h\to 0} \left(nx^{n-1} + \binom{n}{2}x^{n-2}h + \cdots + h^{n-1}\right) \\
&= nx^{n-1} \quad \blacksquare
\end{aligned}$$

Trigonometric Functions

To figure out the derivatives of $\sin x$ and $\cos x$ by the definition, we need these two limits:

$\lim\limits_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ and $\lim\limits_{\theta\to 0} \frac{\cos\theta - 1}{\theta} = 0$

These two limits can be proven using a geometric argument. Then we can find the derivatives of $\sin x$ and $\cos x$ as follows:

$$\begin{aligned}
\frac{d}{dx}(\sin x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x\,(\cos h - 1) + \cos x \sin h}{h} \\
&= \lim_{h\to 0} \left(\sin x \cdot \frac{\cos h - 1}{h} + \cos x \cdot \frac{\sin h}{h}\right) \\
&= \sin x \cdot 0 + \cos x \cdot 1 \\
&= \cos x
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}(\cos x) &= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} \\
&= \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\
&= \lim_{h\to 0} \frac{\cos x\,(\cos h - 1) - \sin x \sin h}{h} \\
&= \lim_{h\to 0} \left(\cos x \cdot \frac{\cos h - 1}{h} - \sin x \cdot \frac{\sin h}{h}\right) \\
&= \cos x \cdot 0 - \sin x \cdot 1 \\
&= -\sin x
\end{aligned}$$

But what about $\tan x$ and the rest of the gang?

Ex 1.

Prove that the derivative of $y = \tan x$ is $\frac{dy}{dx} = \sec^2 x$ by using the derivatives of $\sin x$ and $\cos x$.

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{(\cos x)^2} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x \quad \blacksquare$$

Exponential Functions

To find the derivative of e^x by the definition, we need the following limit:

$$\lim_{h\to 0}\frac{e^h-1}{h}=1$$

This limit is sometimes taken to just be definitional. That is, e is the number where $\lim_{h\to 0}\frac{e^h-1}{h}=1$.

Then,

$$\begin{aligned}\frac{d}{dx}(e^x) &= \lim_{h\to 0}\frac{e^{x+h}-e^x}{h}\\ &= \lim_{h\to 0}\frac{e^x e^h-e^x}{h}\\ &= \lim_{h\to 0}\frac{e^x(e^h-1)}{h}\\ &= \lim_{h\to 0} e^x\cdot\frac{e^h-1}{h}\\ &= e^x\cdot 1\\ &= e^x\end{aligned}$$

Here's how to get the derivative of $y=a^x$:

$$\frac{d}{dx}(a^x)=\frac{d}{dx}\left(\left(e^{\ln a}\right)^x\right)=\frac{d}{dx}\left(e^{x\ln a}\right)=e^{x\ln a}\ln a=a^x\ln a$$

Logarithmic Functions

To find the derivative of logarithms, we can use implicit differentiation.

Ex 2.

Prove that the derivative of $y=\ln x$ is $\frac{dy}{dx}=\frac{1}{x}$.

$$e^y = e^{\ln x}$$

$$e^y = x$$

$$\frac{d}{dx}(e^y)=\frac{d}{dx}(x)$$

$$e^y\cdot\frac{dy}{dx}=1$$

$$\frac{dy}{dx}=\frac{1}{e^y}=\frac{1}{e^{\ln x}}=\frac{1}{x} \quad \blacksquare$$

Note: We can extend our results to include negative x-values: $\frac{d}{dx}(\ln|x|)=\frac{1}{x}$ $(x\neq 0)$

Here's how to get the derivative of $y=\log_a x$.

$$\frac{d}{dx}(\log_a x)=\frac{d}{dx}\left(\frac{\ln x}{\ln a}\right)=\frac{1}{\ln a}\cdot\frac{d}{dx}(\ln x)=\frac{1}{\ln a}\cdot\frac{1}{x}=\frac{1}{x\ln a}$$

Inverse Trigonometric Functions

We can use implicit differentiation with inverse trig functions, too.

Ex 3.

Prove that the derivative of $y = \sin^{-1} x$ is $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$.

$$\sin y = \sin(\sin^{-1} x)$$

$$\sin y = x$$

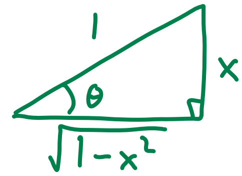
$$\frac{d}{dx}(\sin y) = \frac{d}{dx}(x)$$

$$\cos y \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\cos(\underbrace{\sin^{-1} x}_{\theta})} = \frac{1}{\sqrt{1-x^2}} \quad \blacksquare$$

$$\sin^{-1} x = \theta$$

$$\sin \theta = x = \frac{x}{1}$$



Hyperbolic Functions

Recall that $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$.

Ex 4.

Prove that the derivative of $y = \sinh x$ is $\frac{dy}{dx} = \cosh x$.

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right)$$

$$\frac{d}{dx}(e^{-x}) = e^{-x} \cdot (-1) = -e^{-x}$$

$$= \frac{e^x + e^{-x}}{2}$$

$$= \cosh x \quad \blacksquare$$

Here are the derivatives you might be asked to prove:

$\frac{d}{dx}(\tan x) = \sec^2 x$ $\qquad$ $\frac{d}{dx}(\sec x) = \sec x \tan x$

$\frac{d}{dx}(\cot x) = -\csc^2 x$ $\qquad$ $\frac{d}{dx}(\csc x) = -\csc x \cot x$

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ $\qquad$ $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$ $\qquad$ $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

$\frac{d}{dx}(\ln x) = \frac{1}{x} \quad (x > 0)$

$\frac{d}{dx}(\sinh x) = \cosh x$ $\qquad$ $\frac{d}{dx}(\cosh x) = \sinh x$ $\qquad$ $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$