

Proving Derivatives

(covers parts of Stewart 3.1, 3.3, 3.5, 3.6)

The Power Rule

Here's the proof of the derivative of $f(x) = x^n$:

Suppose $f(x) = x^n$. Then,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h} \\
 &= \lim_{h \rightarrow 0} \frac{x^n + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \dots + h^n - x^n}{h} \\
 &= \lim_{h \rightarrow 0} \frac{nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \dots + h^n}{h} \\
 &= \lim_{h \rightarrow 0} \left(nx^{n-1} + \binom{n}{2}x^{n-2}h + \dots + h^{n-1} \right) \\
 &= nx^{n-1} \quad \blacksquare
 \end{aligned}$$

Trigonometric Functions

To figure out the derivatives of $\sin x$ and $\cos x$ by the definition, we need these two limits:

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \quad \text{and} \quad \lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta} = 0$$

These two limits can be proven using a geometric argument. Then we can find the derivatives of $\sin x$ and $\cos x$ as follows:

$$\begin{aligned}
 \frac{d}{dx}(\sin x) &= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sin x (\cos h - 1) + \cos x \sin h}{h} \\
 &= \lim_{h \rightarrow 0} \left(\sin x \cdot \frac{\cos h - 1}{h} + \cos x \cdot \frac{\sin h}{h} \right) \\
 &= \sin x \cdot 0 + \cos x \cdot 1 \\
 &= \cos x
 \end{aligned}$$

$$\begin{aligned}
 \frac{d}{dx}(\cos x) &= \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cos x (\cos h - 1) - \sin x \sin h}{h} \\
 &= \lim_{h \rightarrow 0} \left(\cos x \cdot \frac{\cos h - 1}{h} - \sin x \cdot \frac{\sin h}{h} \right) \\
 &= \cos x \cdot 0 - \sin x \cdot 1 \\
 &= -\sin x
 \end{aligned}$$

But what about $\tan x$ and the rest of the gang?

Ex 1.

Prove that the derivative of $y = \tan x$ is $\frac{dy}{dx} = \sec^2 x$ by using the derivatives of $\sin x$ and $\cos x$.

Exponential Functions

To find the derivative of e^x by the definition, we need the following limit:

$$\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$$

This limit is sometimes taken to just be definitional. That is, e is the number where $\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$.

Then,

$$\begin{aligned} \frac{d}{dx}(e^x) &= \lim_{h \rightarrow 0} \frac{e^{x+h} - e^x}{h} \\ &= \lim_{h \rightarrow 0} \frac{e^x e^h - e^x}{h} \\ &= \lim_{h \rightarrow 0} \frac{e^x(e^h - 1)}{h} \\ &= \lim_{h \rightarrow 0} e^x \cdot \frac{e^h - 1}{h} \\ &= e^x \cdot 1 \\ &= e^x \end{aligned}$$

Here's how to get the derivative of $y = a^x$:

$$\frac{d}{dx}(a^x) = \frac{d}{dx}((e^{\ln a})^x) = \frac{d}{dx}(e^{x \ln a}) = e^{x \ln a} \ln a = a^x \ln a$$

Logarithmic Functions

To find the derivative of logarithms, we can use implicit differentiation.

Ex 2.

Prove that the derivative of $y = \ln x$ is $\frac{dy}{dx} = \frac{1}{x}$.

Note: We can extend our results to include negative x -values: $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$ ($x \neq 0$)

Here's how to get the derivative of $y = \log_a x$.

$$\frac{d}{dx}(\log_a x) = \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) = \frac{1}{\ln a} \cdot \frac{d}{dx}(\ln x) = \frac{1}{\ln a} \cdot \frac{1}{x} = \frac{1}{x \ln a}$$

Inverse Trigonometric Functions

We can use implicit differentiation with inverse trig functions, too.

Ex 3.

Prove that the derivative of $y = \sin^{-1} x$ is $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$.

Hyperbolic Functions

Recall that $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$.

Ex 4.

Prove that the derivative of $y = \sinh x$ is $\frac{dy}{dx} = \cosh x$.

Here are the derivatives you might be asked to prove:

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \quad (x > 0)$$

$$\frac{d}{dx}(\sinh x) = \cosh x$$

$$\frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$