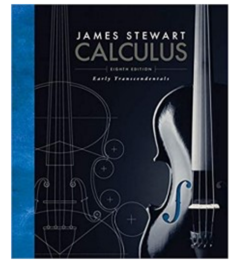
Definition of the Derivative

(covers parts of Stewart 2.1, 2.7, and 2.8)



Ex 1.

Suppose you drop a calculus book from the top of building 61. And suppose you figure out that the distance fallen is $y = 16t^2$ (y is in feet, and t is in seconds). What is the average speed of the book between $t = 1$ sec and $t = 2$ sec?

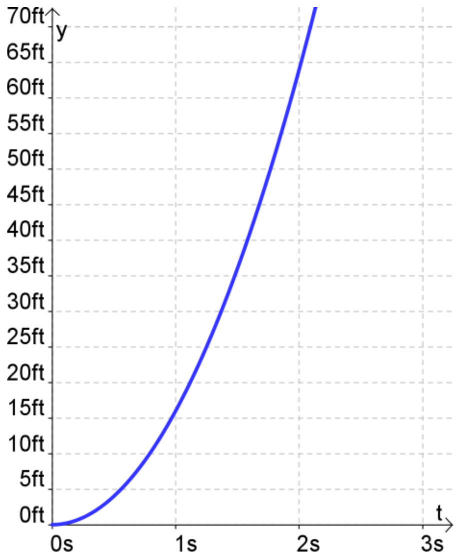
What is the average speed of the book between $t = 1$ sec and $t = 1.1$ sec?

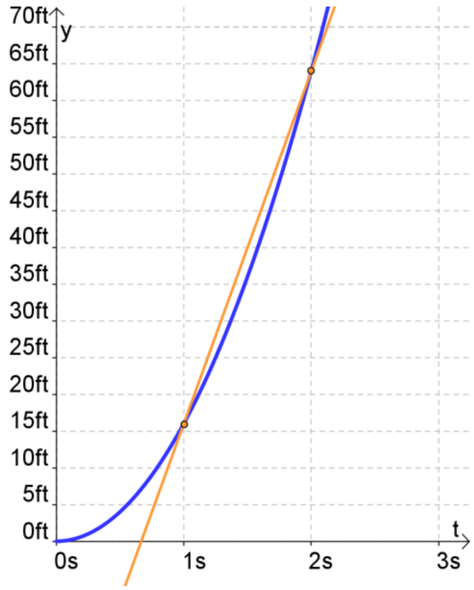
What is the speed of the book at $t = 1$ second? (This is called the *instantaneous* speed at $t = 1$.)

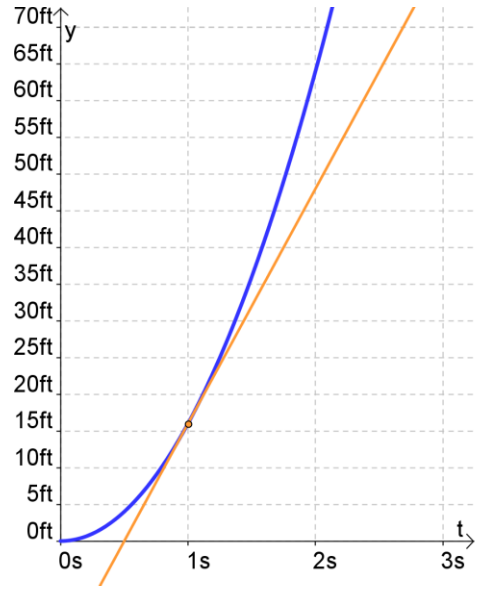
Length of time interval	Average speed over interval

Instantaneous speed: ______________

We can graph the distance the book has traveled with respect to time.







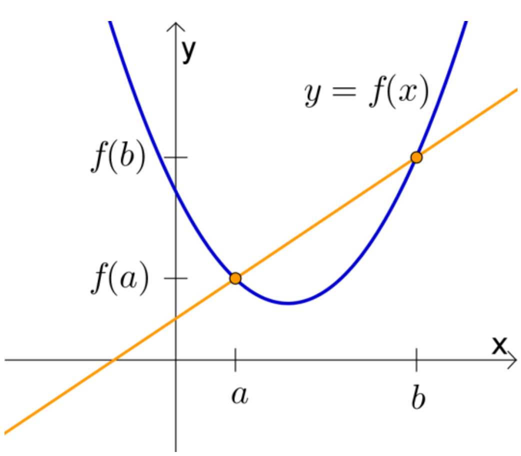
Graphically, an **average speed** is the _________ of a __________________________!
(ex: 200 miles in 4 hours is 50 mph.)

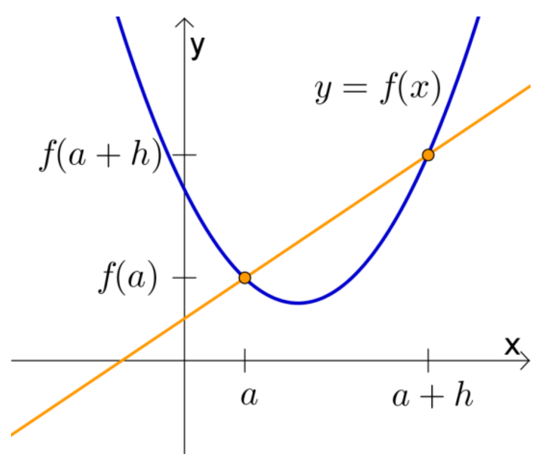
Instantaneous speed is the _________ of a _____________________________!
(ex: I'm going 60 mph right now.)

(Note that the average speeds get closer to the instantaneous speed when the time interval shrinks.)

In general, the ______________________________________ of $y = f(x)$ with respect to x over the interval $[a, b]$ is:

$$\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a} = \frac{f(a + h) - f(a)}{h}$$





As h (our interval width) approaches 0, the slopes of the secant lines approach the slope of the tangent line. With limits, we write the tangent slope like this:

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

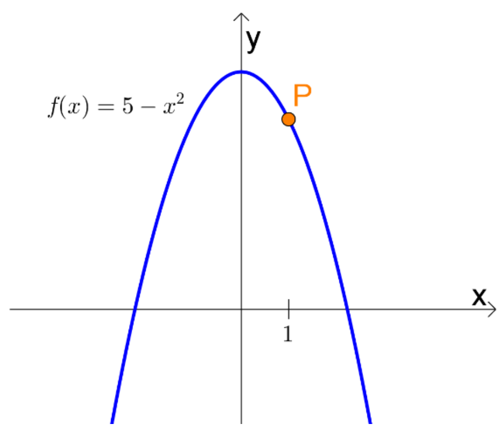
The name given to this particular limit is the _______________ of f at the point a. It is written $f'(a)$, and is read "f prime of a".

So, $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ has 3 interpretations:

1. The **instantaneous rate of change** of f with respect to x at $x=a$.
2. The **slope of the tangent line** at $x=a$.
3. The **derivative** $f'(a)$.

Ex 2.

Find the slope of the tangent line to the curve $f(x)=5-x^2$ at the point $P(1,4)$.



Now find an equation of the tangent line at P.

Find $f'(1)$. __________

Find the rate at which $f(x)$ is changing with respect to x at $x=1$. ___________

The Derivative as a Function

We can make a derivative function by letting the x-value be a variable, rather than a specific $x = a$.

$$\boldsymbol{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

f is ______________________ at x if $f'(x)$ exists.

____________________ is the process of calculating the derivative.

The following directions are all the same:
"Differentiate $f(x) = x^2$."
"Find the derivative of $f(x) = x^2$."
"Find $f'(x)$ when $f(x) = x^2$."
"Find the slope of the tangent line of $f(x) = x^2$."
"Find the instantaneous rate of change of $f(x) = x^2$."

Ex 3.
Find the derivative of $f(x) = x^3$ using the limit definition.

Ex 4.

Find the derivative of $f(x) = \frac{x}{x-3}$ using the limit definition.

What is the slope of the tangent to the curve $y = \frac{x}{x-3}$ at $x = 2$?

Find the rate at which $f(x) = \frac{x}{x-3}$ is changing with respect to x at $x = 2$. ________

Ex 5.

Find the derivative of $f(x) = \sqrt{x^2 + 1}$ using the limit definition.

Now find the tangent line to the curve $y = \sqrt{x^2 + 1}$ at $x = 1$.

Note: There are many ways people write the derivative of $y = f(x)$:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = D(f)(x) = D_x f(x)$$

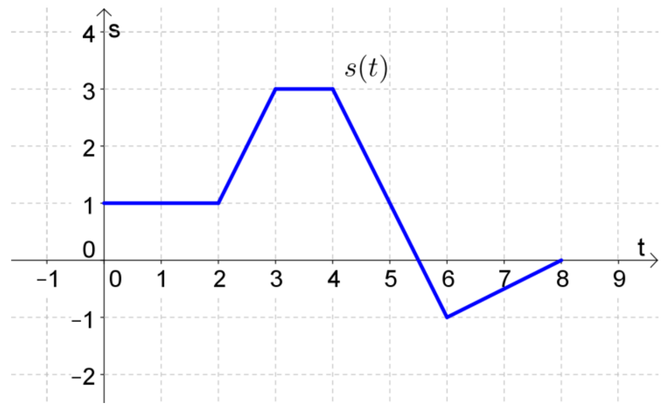
And here's what it looks like to plug a value into the derivative:

$$f'(2) = \left.\frac{dy}{dx}\right|_{x=2}$$

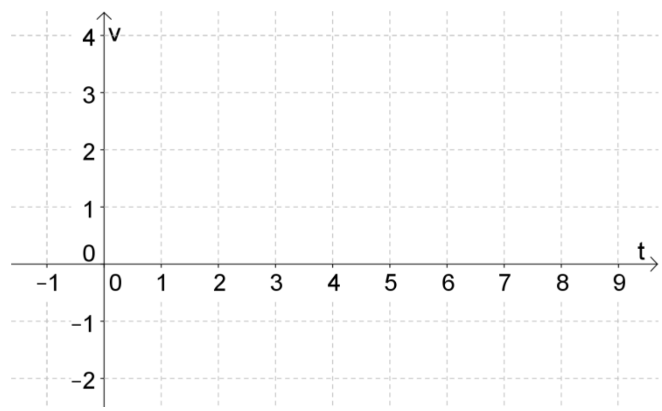
One way to help ground your understanding of the derivative is to think about position vs. velocity.

Note that velocity is the rate at which position changes. That is, $v(t) = s'(t)$.
This also means that $v(t)$ is the slope of the tangent line of $s(t)$.

Ex 6.
Suppose a robot moves back and forth along a line, and that the position of the robot (in meters) over time (in seconds) is given by the function $s(t)$ to the right.



a. Graph the velocity function $v(t)$.



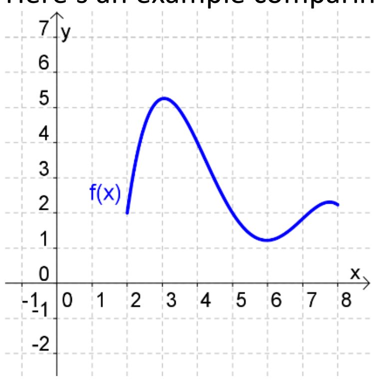
b. Find $s'(1)$. __________

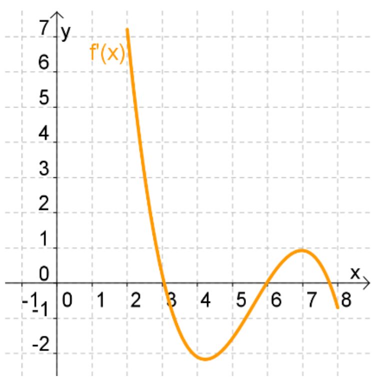
c. Find $s'(5)$. __________

d. Find $s'(6)$. __________

e. Find $v(7)$. __________

Here's an example comparing $f(x)$ and $f'(x)$.

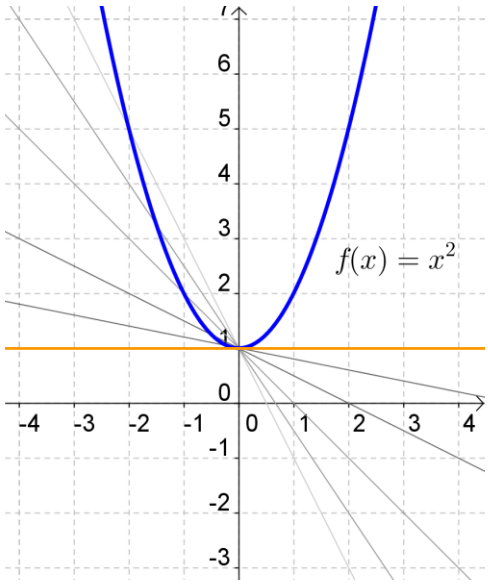


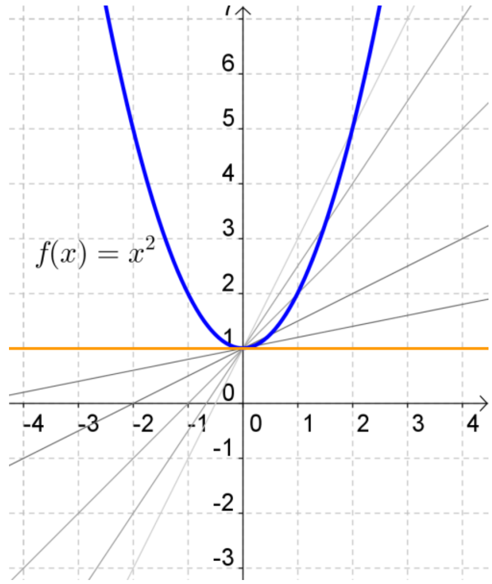


Note that $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ only exists if both of the following limits exist and are equal:

$$\lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h} \qquad \text{and} \qquad \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$$

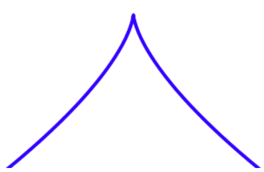
Left-hand derivative at $x = a$ **Right-hand derivative at $x = a$**



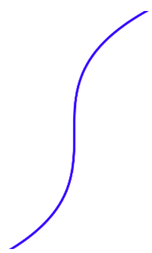


So, derivatives won't exist at the following kinds of places:

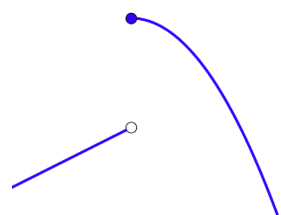
corner



cusp



vertical tangent



discontinuity

Theorem: If f has a derivative at $x = a$, then f is continuous at $x = a$.
(That is, differentiable functions are continuous. And if a function is not continuous at a point, then it is not differentiable there.)

Proof: Suppose that f is differentiable at $x = a$. Then $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = L$ for some real number L.
Also, $\lim_{x\to a}[f(x) - f(a)] = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \cdot (x-a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \cdot \lim_{x\to a}(x-a) = L \cdot 0 = 0.$
Thus, we have: $\lim_{x\to a} f(x) = \lim_{x\to a}\left[f(a) + \big(f(x) - f(a)\big)\right] = \lim_{x\to a} f(a) + \lim_{x\to a}\big(f(x) - f(a)\big) = f(a)$
And so, f is continuous at $x = a$. ■