

Limits (Numeric and Algebraic)

(covers parts of Stewart 2.2, 2.3, and 2.6)

We can find limits given the graph of a function. But what if we don't know the graph?

One way to estimate the limit is numerically (that is, plug in numbers).

Ex 1.

Find $\lim_{x \rightarrow 3^+} \frac{x+2}{x-3}$

x	3.2	3.1	3.01	3.001	3.0001	→	3
$x + 2$	5.2	5.1	5.01	5.001	5.0001	→	
$x - 3$	0.2	0.1	0.01	0.001	0.0001	→	
$\frac{x + 2}{x - 3}$	26	51	501	5001	50001	→	

Ex 2.

Find $\lim_{x \rightarrow 2^-} \frac{5-x}{x-2}$

x	1.8	1.9	1.99	1.999	1.9999	→	2
$5 - x$	3.2	3.1	3.01	3.001	3.0001	→	
$x - 2$	-0.2	-0.1	-0.01	-0.001	-0.0001	→	
$\frac{5 - x}{x - 2}$	-16	-31	-301	-3001	-30001	→	

Ex 3.

Find $\lim_{x \rightarrow -4^-} \frac{1-2x}{x+4}$

x	-4.2	-4.1	-4.01	-4.001	-4.0001	→	-4
$1 - 2x$	9.4	9.2	9.02	9.002	9.0002	→	
$x + 4$	-0.2	-0.1	-0.01	-0.001	-0.0001	→	
$\frac{1 - 2x}{x + 4}$	-47	-92	-902	-9002	-90002	→	

Making tables of numbers is cumbersome. Let's transition to doing the analysis without a table.

Ex 4.

$$\text{Find } \lim_{x \rightarrow 3^-} \frac{2x+1}{x-3}$$

Ex 5.

$$\text{Find } \lim_{x \rightarrow -1^+} \frac{2+3x}{x+1}$$

Ex 6.

$$\text{Find } \lim_{x \rightarrow 2^-} \frac{2+3x}{x+1}$$

Ex 7.

$$\text{Find } \lim_{x \rightarrow (\pi/2)^-} e^{\sec x}$$

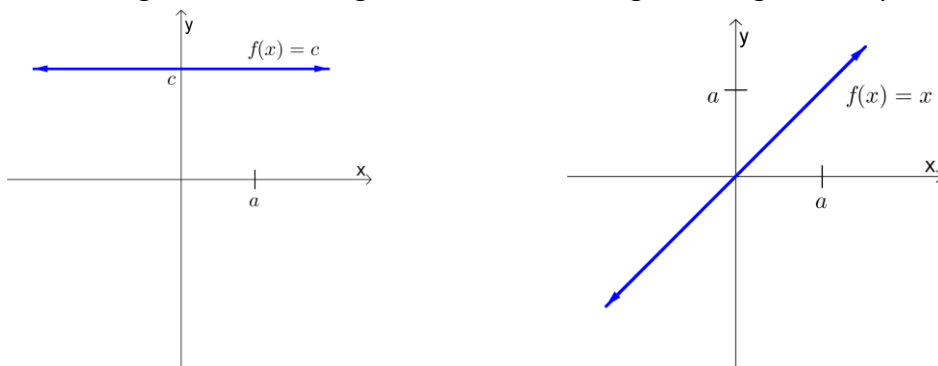
Ex 8.

$$\text{Find } \lim_{x \rightarrow -2^+} \ln(4 - x^2)$$

Ex 9.

$$\text{Find } \lim_{x \rightarrow \infty} \frac{2}{\tan^{-1}(x-10)}$$

Now let's get some building blocks for evaluating limits algebraically.



Note: In general, $\lim_{x \rightarrow a} c = \underline{\hspace{2cm}}$ and $\lim_{x \rightarrow a} x = \underline{\hspace{2cm}}$.

ex: $\lim_{x \rightarrow 71} 3 = \underline{\hspace{2cm}}$ $\lim_{x \rightarrow 42} x = \underline{\hspace{2cm}}$

The Limit Laws

If $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ both exist, then the following laws are true:

1. $\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$
2. $\lim_{x \rightarrow a} c f(x) = c \lim_{x \rightarrow a} f(x)$ (here c is a constant)
3. $\lim_{x \rightarrow a} [f(x)g(x)] = \left[\lim_{x \rightarrow a} f(x) \right] \left[\lim_{x \rightarrow a} g(x) \right]$
4. $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$ (if $\lim_{x \rightarrow a} g(x) \neq 0$)
5. $\lim_{x \rightarrow a} [f(x)]^p = \left[\lim_{x \rightarrow a} f(x) \right]^p$ (if $\left[\lim_{x \rightarrow a} f(x) \right]^p$ exists)

ex: Find $\lim_{x \rightarrow -2} (2x^3 - x^2 + 3)$ using the Limit Laws.

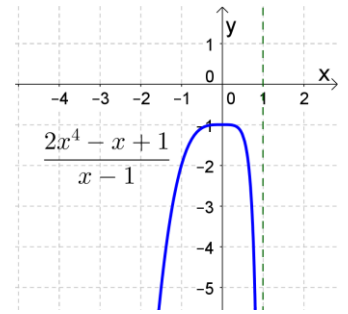
$$\begin{aligned}
 \lim_{x \rightarrow -2} (2x^3 - x^2 + 3) &= \lim_{x \rightarrow -2} 2x^3 - \lim_{x \rightarrow -2} x^2 + \lim_{x \rightarrow -2} 3 \\
 &= 2 \lim_{x \rightarrow -2} x^3 - \lim_{x \rightarrow -2} x^2 + \lim_{x \rightarrow -2} 3 \\
 &= 2 \left[\lim_{x \rightarrow -2} x \right]^3 - \left[\lim_{x \rightarrow -2} x \right]^2 + \lim_{x \rightarrow -2} 3 \\
 &= 2 \cdot (-2)^3 - (-2)^2 + 3 \\
 &= -16 - 4 + 3 \\
 &= -17
 \end{aligned}$$

Note: If $P(x)$ and $Q(x)$ are polynomials, then...

- $\lim_{x \rightarrow a} P(x) = P(a)$
- $\lim_{x \rightarrow a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)}$ (if $Q(a) \neq 0$)

Ex 10.

Find $\lim_{x \rightarrow -1} \frac{2x^4 - x + 1}{x - 1}$



What happens when both the numerator and denominator go to 0?

Consider $f(x) = \frac{x^2-1}{x-1}$. What is $f(1)$? _____

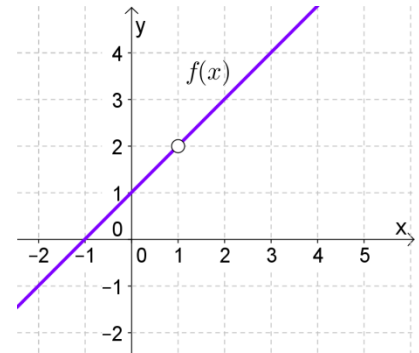
Now what happens to $f(x)$ as x gets close to 1?

x	0.8	0.99	0.999	1	1.001	1.01	1.2
$f(x)$							

It looks like $f(x)$ approaches _____ as x approaches 1.

In other words: $\lim_{x \rightarrow 1} \frac{x^2-1}{x-1} = 2$

To the right is a picture of $f(x) = \frac{x^2-1}{x-1}$ looks like.

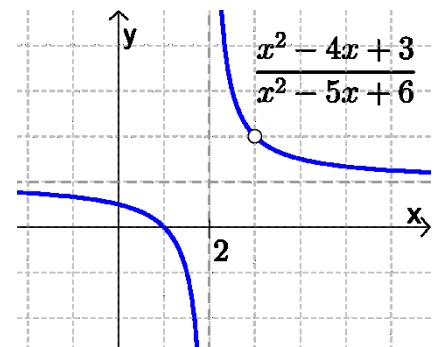


Note that we can figure the above limit out algebraically, too:

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = \lim_{x \rightarrow 1} \frac{(x + 1)(x - 1)}{x - 1} = \lim_{x \rightarrow 1} (x + 1) = 1 + 1 = 2$$

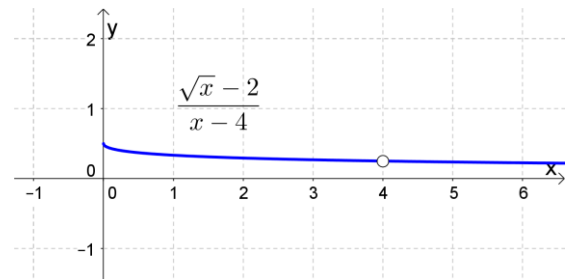
Ex 11.

$\lim_{x \rightarrow 3} \frac{x^2 - 4x + 3}{x^2 - 5x + 6}$



Ex 12.

Find $\lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{x - 4}$

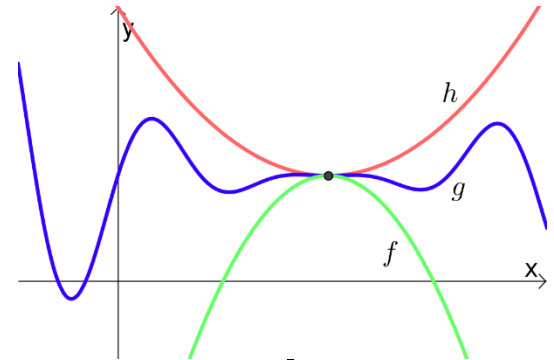


The Squeeze Theorem

Suppose that...

1. $f(x) \leq g(x) \leq h(x)$ for all x in some open interval containing a , except possibly at $x = a$ itself, and
2. $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$.

Then $\lim_{x \rightarrow a} g(x) = L$.

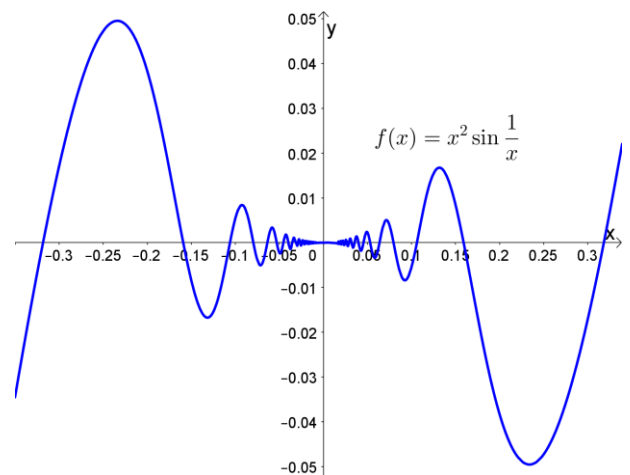


Note: The Squeeze Theorem has other names: Sandwich Theorem, Pinching Theorem, Two Policemen and a Drunk Theorem, etc.



Ex 13.

Use the Squeeze Theorem to prove that $\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x} = 0$.



Notes:

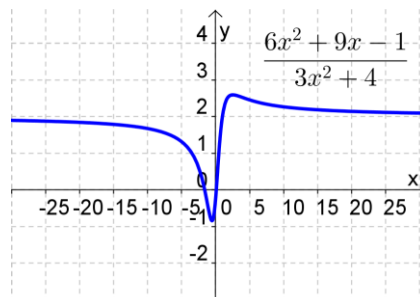
$$\lim_{x \rightarrow \pm\infty} k = \underline{\quad} \quad (k \text{ is a constant})$$

$$\lim_{x \rightarrow \pm\infty} \frac{1}{x} = \underline{\quad}$$

$$\lim_{x \rightarrow \pm\infty} \frac{1}{x^2} = \underline{\quad}$$

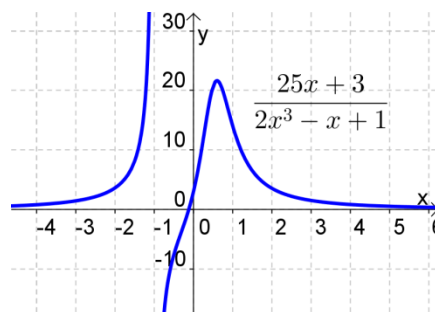
Ex 14.

$$\lim_{x \rightarrow \infty} \frac{6x^2 + 9x - 1}{3x^2 + 4}$$



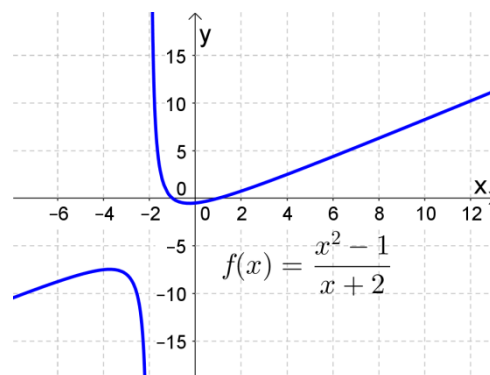
Ex 15.

$$\lim_{x \rightarrow -\infty} \frac{25x + 3}{2x^3 - x + 1}$$



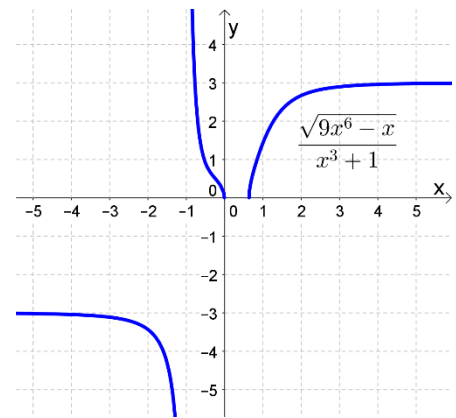
Ex 16.

$$\lim_{x \rightarrow \infty} \frac{x^2 - 1}{x + 2}$$

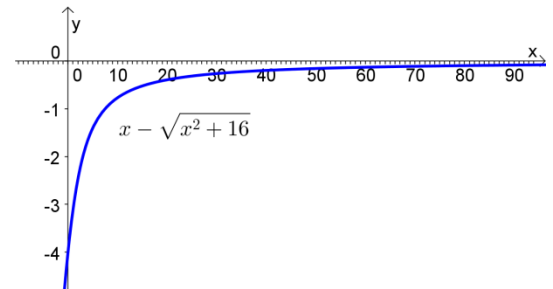


Ex 17.

$$\lim_{x \rightarrow \infty} \frac{\sqrt{9x^6 - x}}{x^3 + 1}$$

 $\infty - \infty$ Limits**Ex 18.**

$$\lim_{x \rightarrow \infty} (x - \sqrt{x^2 + 16})$$



Ex 19.

Find the horizontal asymptote of $y = \frac{\sin x}{x}$ and use the Squeeze Theorem to prove it is a horizontal asymptote.

