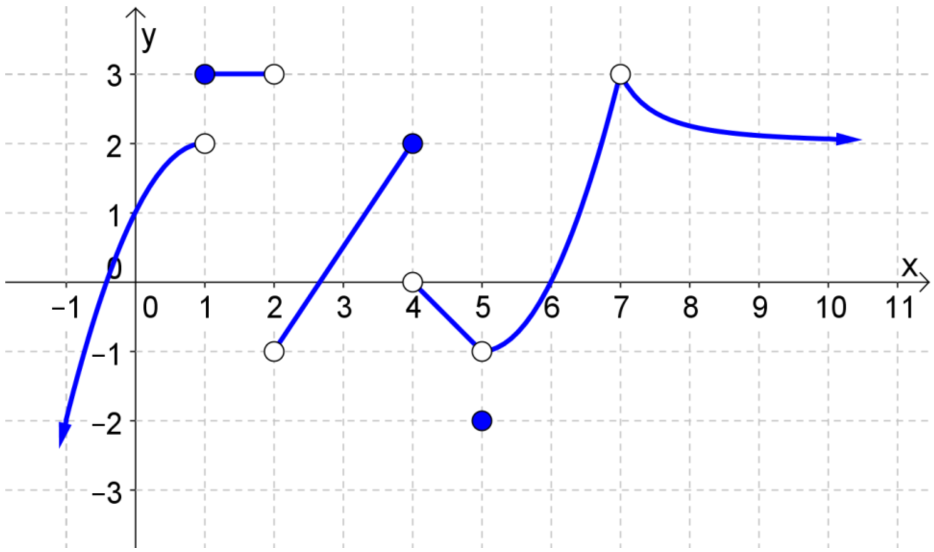
Limits (Visual)

(covers parts of Stewart 2.2 and 2.6)

Let's warm up by visually plugging inputs into a function and reading off the outputs.



Ex 1.
Find the following, given the graph of the crazy piecewise-defined function, $f(x)$, above.

$f(0)$ $f(1)$ $f(2)$

$f(4)$ $f(5)$ $f(6)$

$f(7)$

The **limit** of a function is a core concept in calculus. Other calculus concepts we'll explore (namely, the derivative and the integral) are defined in terms of limits. To get the intuition behind what a limit means, let's first approach limits visually. Here, you have to imagine an animation of the inputs and outputs of a function. Ask: what's happening to the outputs as the inputs are changing?
Note: $\boldsymbol{x \to 1^-}$ means "**$\boldsymbol{x}$ approaches 1 from the left**"
$\boldsymbol{x \to 1^+}$ means "**$\boldsymbol{x}$ approaches 1 from the right**"

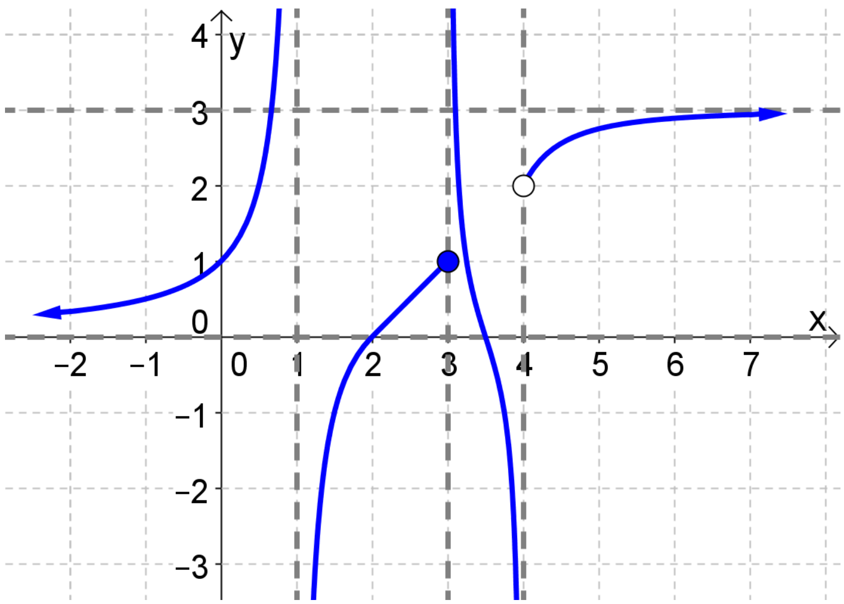
Ex 2.
Find the following limits, given the graph of $f(x)$ above.

$\lim_{x\to 1^-} f(x)$ $\lim_{x\to 1^+} f(x)$ $\lim_{x\to 2^-} f(x)$ $\lim_{x\to 2^+} f(x)$

$\lim_{x\to 4^-} f(x)$ $\lim_{x\to 4^+} f(x)$ $\lim_{x\to 5^-} f(x)$ $\lim_{x\to 5^+} f(x)$

$\lim_{x\to 6^-} f(x)$ $\lim_{x\to 6^+} f(x)$ $\lim_{x\to 7^-} f(x)$ $\lim_{x\to 7^+} f(x)$

We can use $+\infty$ and $-\infty$ to describe the behavior of the following function.



Ex 3.
Find the following limits, given the graph of $f(x)$ above.

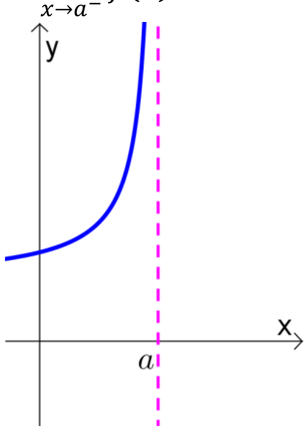
$\lim_{x\to 1^-} f(x)$ $\quad\lim_{x\to 1^+} f(x)$ $\quad\lim_{x\to 3^-} f(x)$ $\quad\lim_{x\to 3^+} f(x)$

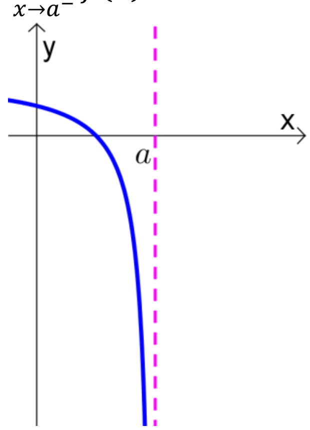
$\lim_{x\to 4^-} f(x)$ $\quad\lim_{x\to 4^+} f(x)$ $\quad\lim_{x\to \infty} f(x)$ $\quad\lim_{x\to -\infty} f(x)$

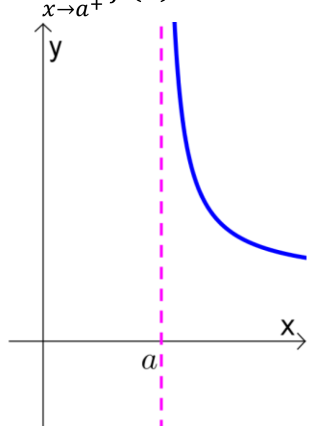
Limits with $x \to c^-$ or $x \to c^+$ are called "one-sided" limits, because the inputs (x-values) are approaching the number c from one side (either the left or right side).

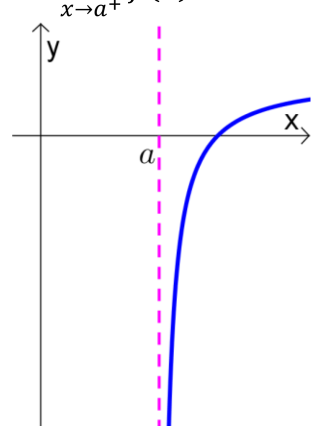
Note that if a one-sided limit is ∞ or $-\infty$, then you have a **vertical asymptote**.
Here are the four ways that could happen:

$\lim_{x\to a^-} f(x) = \infty$ or $\lim_{x\to a^-} f(x) = -\infty$ or $\lim_{x\to a^+} f(x) = \infty$ or $\lim_{x\to a^+} f(x) = -\infty$









Let's look at some familiar functions that have vertical asymptotes.

Ex 4.
Find the following limits.

$\lim_{x \to \frac{\pi}{2}^-} \tan x$

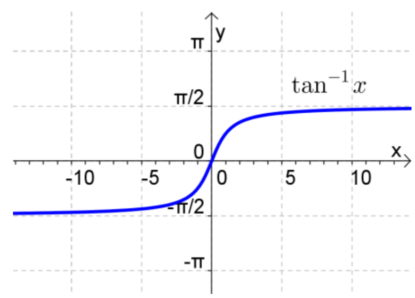
$\lim_{x \to 0^+} \ln x$

$\lim_{x \to 0^+} \frac{1}{x}$

$\lim_{x \to 0^-} \frac{1}{x}$

Also note that if $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$, then $f(x)$ has a **horizontal asymptote** $y = L$.

For example, $\lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2}$, so $\tan^{-1} x$ has a horizontal asymptote $y = \frac{\pi}{2}$.



Ex 5.
Find the following limits.

$\lim_{x \to \infty} \frac{1}{x}$

$\lim_{x \to -\infty} e^x$

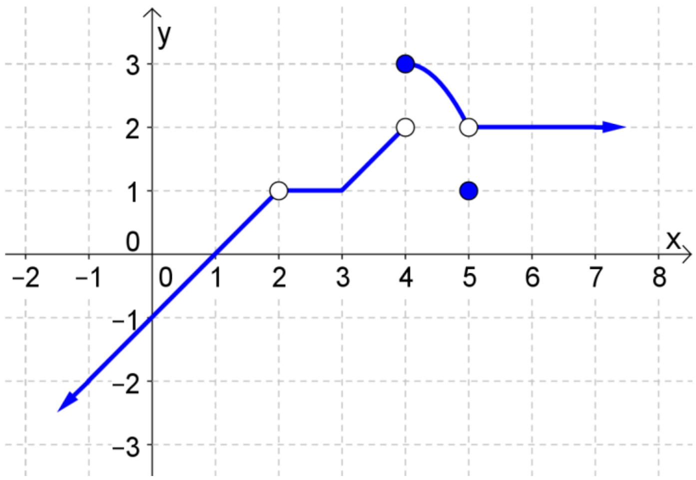
$\lim_{x \to -\infty} 2 - e^{x+1}$

$\lim_{x \to \infty} 2 - e^{x+1}$

In the future, we'll mostly use the "regular" limit, which requires the left- and right-hand limits to exist and be equal. The "regular" limit does not have a "-" or "+", and is just written using $x \to c$.

Note: $\lim_{x \to c} f(x) = L$ if an only if $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L$

As with one-sided limits, regular limits only care about the behavior of the function *near* $x = c$, not *at* $x = c$.



Ex 6.
Find the following limits, given the graph of $f(x)$ above.

$\lim_{x \to 2^-} f(x)$ $\lim_{x \to 2^+} f(x)$ $\lim_{x \to 2} f(x)$

$\lim_{x \to 3^-} f(x)$ $\lim_{x \to 3^+} f(x)$ $\lim_{x \to 3} f(x)$

$\lim_{x \to 4^-} f(x)$ $\lim_{x \to 4^+} f(x)$ $\lim_{x \to 4} f(x)$

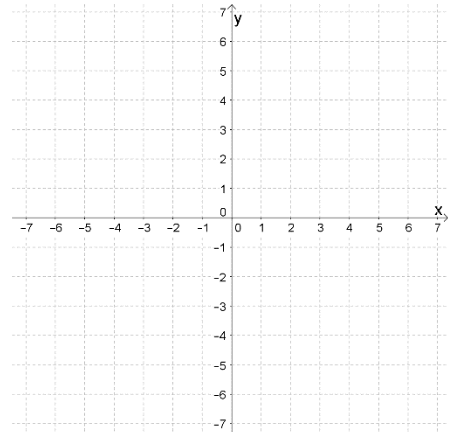
$\lim_{x \to 5^-} f(x)$ $\lim_{x \to 5^+} f(x)$ $\lim_{x \to 5} f(x)$

$\lim_{x \to 1} f(x)$ $\lim_{x \to 0} f(x)$ $\lim_{x \to \infty} f(x)$ $\lim_{x \to -\infty} f(x)$

Let's review how piecewise-defined functions look algebraically.

Ex 7.

Suppose $f(x) = \begin{cases} x+1 & \text{if } x < -2 \\ 4 & \text{if } -2 \le x < 1 \\ 5 & \text{if } x = 1 \\ (x-3)^2 & x > 1 \end{cases}$. Find the following limits.



$\lim_{x\to -2^-} f(x)$ $\lim_{x\to -2^+} f(x)$ $\lim_{x\to -2} f(x)$

$\lim_{x\to 1^-} f(x)$ $\lim_{x\to 1^+} f(x)$ $\lim_{x\to 1} f(x)$

In the next example, we're given information about a function, and we have to construct its graph.

Ex 8.

Draw the graph of a function $f(x)$, where $\lim_{x\to -1^-} f(x) = \infty$, $\lim_{x\to -1^+} f(x) = -\infty$, $\lim_{x\to 2} f(x) = 4$, $f(2) = 3$, $\lim_{x\to -\infty} f(x) = 1$, and $\lim_{x\to \infty} f(x) = -\infty$.