Math 180 – Final Exam Info and Review Exercises

Fall 2017, Prof. Beydler

Test Info

- Date: Tuesday, December 12, 2017 from 1:30pm-4:00pm
- Will cover almost all packets.
- You'll need a **scientific calculator** for the final exam.
- No notes, no books, no phones during the final exam. Please don't fail the class because of a phone in your lap!
- As usual, there will be a seating chart for the final exam.
- Where to get help as you're studying:
  - Office hours
  - TMARC, LAC, or other tutoring centers
  - E-mail me at dbeydler@mtsac.edu
- If you go to the TMARC/LAC for 4 hours between Test #3 and the Final Exam, you'll get 1% extra credit towards the Final Exam. I'll be at tutoring events on Friday, 12/1, 2pm-4pm in 61-3414, and Saturday, 12/9, 11am-3pm in 61-3419. Those tutoring hours will count, too!

Not on the final exam:

- Packet #10 (about proving derivatives)
- Packet #11 (about linearization and differentials)

Formulas and stuff

(Note: Know all of these except for the ones with 🏝 next to them, which I'll give you. This list is not meant to include everything you'll need to know on the test.)

Definition of the derivative: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$\frac{d}{dx}(c) = 0$ $\qquad \frac{d}{dx}(x^n) = nx^{n-1}$ $\qquad (cf)' = cf'$

$(f \pm g)' = f' \pm g'$ $\qquad (fg)' = fg' + gf'$ $\qquad \left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$

$\frac{d}{dx}(\sin x) = \cos x$ $\qquad \frac{d}{dx}(\cos x) = -\sin x$ $\qquad \frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{d}{dx}(\sec x) = \sec x \tan x$ $\qquad \frac{d}{dx}(\cot x) = -\csc^2 x$ $\qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ $\qquad \frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$ $\qquad \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

$\frac{d}{dx}(\csc^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$ 🏝 $\qquad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$ 🏝 $\qquad \frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$ 🏝

$\frac{d}{dx}(e^x) = e^x$ $\qquad \frac{d}{dx}(a^x) = a^x \ln a$

$\frac{d}{dx}(\ln x) = \frac{1}{x} \quad (x > 0)$ $\qquad \frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$

$\frac{d}{dx}(\sinh x) = \cosh x$

$\frac{d}{dx}(\cosh x) = \sinh x$

$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

$\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$

$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$

$\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$

$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$

$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$

$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$

$\frac{d}{dx}(\operatorname{csch}^{-1} x) = -\frac{1}{|x|\sqrt{x^2+1}}$

$\frac{d}{dx}(\operatorname{sech}^{-1} x) = -\frac{1}{x\sqrt{1-x^2}}$

$\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2}$

$\sinh x = \frac{e^x - e^{-x}}{2}$

$\cosh x = \frac{e^x + e^{-x}}{2}$

$\tanh x = \frac{\sinh x}{\cosh x}$

$\coth x = \frac{\cosh x}{\sinh x}$

$\operatorname{sech} x = \frac{1}{\cosh x}$

$\operatorname{csch} x = \frac{1}{\sinh x}$

$\cosh^2 x - \sinh^2 x = 1$

Here are some helpful geometry formulas to know for related rates and optimization problems:

Distance/rate/time formula: $d = rt$

Pythagorean Theorem: $a^2 + b^2 = c^2$ (or $(leg)^2 + (leg)^2 = (hypotenuse)^2$)

Area of rectangle: $A = lw$

Area of circle: $A = \pi r^2$

Area of triangle: $A = \frac{1}{2}bh$

Circumference of circle: $C = 2\pi r = \pi d$

How to get perimeter of any polygon (just add the lengths of the sides).

How to get the surface area of a 3-D surface (just add the areas of the faces/sides).

Volume of a box (also called a rectangular prism): $V = lwh$

Volume of circular cylinder: $V = \pi r^2 h$

Surface area of sphere: $S = 4\pi r^2$

Volume of sphere: $V = \frac{4}{3}\pi r^3$

Volume of cone: $V = \frac{1}{3}\pi r^2 h$

Newton's method: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Summation Properties

$$\sum_{i=1}^{n}(a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n} c a_i = c \cdot \sum_{i=1}^{n} a_i$$

$$\sum_{i=1}^{n} c = n \cdot c$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Integral Properties

$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

$\int_a^a f(x)\,dx = 0$

$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

$\int_a^b \big(f(x) \pm g(x)\big)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

$(\min f)\cdot(b-a) \le \int_a^b f(x)\,dx \le (\max f)\cdot(b-a)$

If $f(x) \ge g(x)$ on $[a,b]$, then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$.

The Fundamental Theorem of Calculus, Part 1: $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

The Fundamental Theorem of Calculus, Part 2: $\int_a^b f(x)\,dx = F(b) - F(a)$

Substitution: $\int f\big(g(x)\big)g'(x)\,dx = \int f(u)\,du$

Integration by parts: $\int udv = uv - \int vdu$

Rolle's Theorem

Suppose that

1. $f(x)$ is continuous on $[a,b]$,
2. $f(x)$ is differentiable on (a,b), and
3. $f(a) = f(b)$.

Then there is at least one number c in (a,b) at which $f'(c) = 0$.

The Mean Value Theorem

Suppose that

1. $f(x)$ is continuous on $[a,b]$, and
2. $f(x)$ is differentiable on (a,b).

Then there is at least one point c in (a,b) at which

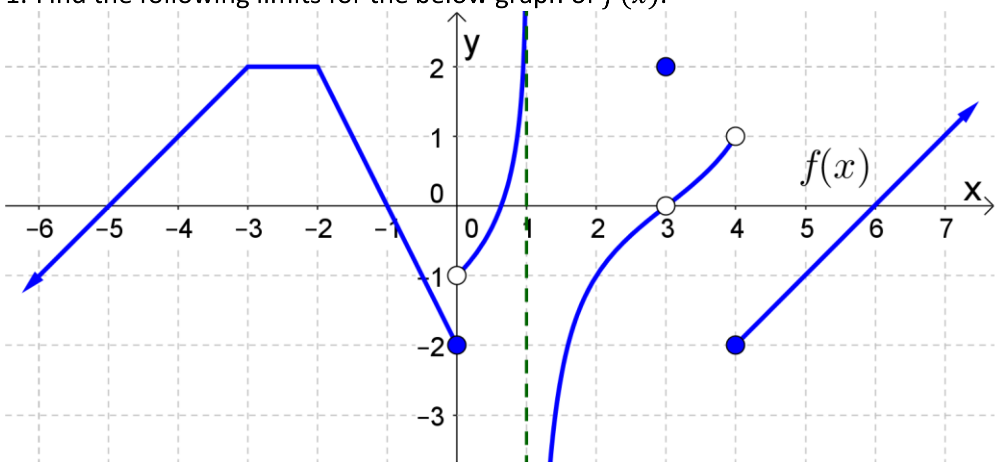
$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

Review Exercises

Note: If you write up the answers to all of the review exercises listed below, and hand them in at the final exam, you can earn up to 3% extra credit towards your final exam percentage (depending on neatness and completeness)! It is important to understand that these review exercises are <u>not guaranteed to cover all</u> of the potential problems on the final exam. Please review the notes, practice problems, previous quizzes, previous tests, previous test review exercises, and homework problems to fully prepare for the final exam.

VISUAL CONCEPTS

1. Find the following limits for the below graph of $f(x)$.



$\lim_{x\to 0^-} f(x)$ $\lim_{x\to 0^+} f(x)$ $\lim_{x\to 0} f(x)$

$\lim_{x\to 1^-} f(x)$ $\lim_{x\to 1^+} f(x)$ $\lim_{x\to 1} f(x)$

$\lim_{x\to 3^-} f(x)$ $\lim_{x\to 3^+} f(x)$ $\lim_{x\to 3} f(x)$

$\lim_{x\to 4^-} f(x)$ $\lim_{x\to 4^+} f(x)$ $\lim_{x\to 4} f(x)$

$\lim_{x\to \infty} f(x)$ $\lim_{x\to -\infty} f(x)$

2. Using the graph of $f(x)$ from the previous question, answer the following.
   a) Is f continuous or discontinuous at $x = -2$?
   b) Is f continuous or discontinuous at $x = 0$?
   c) Is f continuous from the left at $x = 0$?
   d) Is f continuous from the right at $x = 2$?
   e) Is f continuous from the left at $x = 3$?
   f) Is f differentiable at $x = 2$?
   g) Is f differentiable at $x = 3$?
   h) Is f differentiable at $x = 4$?
   i) Find $f'(-4.1)$.
   j) Find $f'(-2.5)$.
   k) Find $f'(-2)$.
   l) Find $f'(-1)$.
   m) Find $\int_{-5}^{-3} f(x)\,dx$.
   n) Find $\int_{-4}^{-1} f(x)\,dx$.
   o) Find $\int_{-3}^{0} f(x)\,dx$.
   p) Find $\int_{5}^{5} f(x)\,dx$.

LIMITS AND CONTINUITY

3. Find the following limits.

a) $\lim\limits_{x\to 1/3} \frac{3x^2+5x-2}{6x^2+x-1}$

b) $\lim\limits_{x\to 2^+} \frac{x+1}{2-x}$

c) $\lim\limits_{x\to 2} \frac{x+1}{2-x}$

d) $\lim\limits_{x\to 2} \frac{x+1}{(2-x)^2}$

e) $\lim\limits_{x\to 0^+} e^{\frac{1}{\pi}\tan^{-1}(\ln x)}$

f) $\lim\limits_{x\to \infty} \frac{-x^3+7\sqrt{x}}{2-3x^2}$

g) $\lim\limits_{x\to -\infty} \frac{\sqrt{x^4+5x}}{x^3+2}$

h) $\lim\limits_{x\to 0} x^4 e^{\cos(1/x^2)}$ (Use the Squeeze Theorem to find/show this one.)

i) $\lim\limits_{x\to 0} \frac{\tan 7x}{\sin 3x}$

j) $\lim\limits_{x\to 0} \frac{1-\cos 3x}{\cos^2 5x-1}$

k) $\lim\limits_{x\to \infty} \frac{e^{4x}-1-4x}{x^2}$

l) $\lim\limits_{x\to 1^+} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$

m) $\lim\limits_{x\to 0^+} x^2 \ln x$

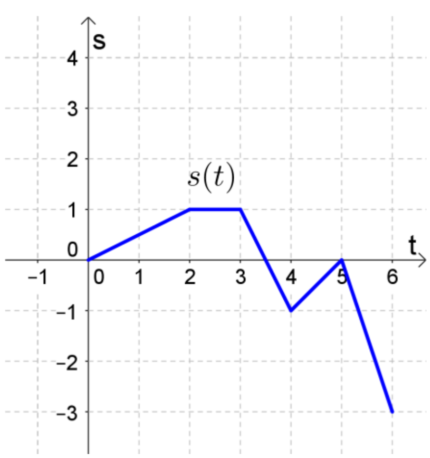
n) $\lim\limits_{x\to 0^+} (\csc x)^{\sin x}$

o) $\lim\limits_{x\to 0^+} (\cos x)^{1/x}$

4. How would you define $f(1)$ in a way that makes $f(x) = \frac{2-x-x^2}{3x^2-x-2}$ continuous at $x = 1$?

DERIVATIVES

5. Suppose a bug moves back and forth along a line, and that the position of the bug over time is given by the function $s(t)$ below. Graph the velocity function $v(t)$.



6. Find the derivatives of the following functions <u>using the limit definition</u>.

a) $f(x) = \sqrt{2x+3}$

b) $f(x) = \frac{1}{3x-1}$

7. Differentiate the following functions.
   a) $f(x) = \frac{3}{x} - 5\ln x + \frac{1}{\sqrt[3]{x^2}} + 2\sqrt{x} - \frac{10}{x^3}$
   b) $y = \tan^{-1}\left(\sqrt{\sin x}\right)$
   c) $f(x) = \sec\left(\frac{e^{2x}}{x^2+3}\right)$
   d) $f(x) = e^{x\cot x}$
   e) $y = \frac{x^3+2x^2-5x-1}{x^2} - x^4\ln 3x$
   f) $f(x) = e^{-2x+3}\cos^5 x$
   g) $y = \frac{\cos x\cdot e^{4\sqrt{x}}}{x\cdot \ln x\cdot \sqrt[3]{x^2+2}}$
   h) $y = (\ln x)^{\tan x}$
   i) $y = \ln\left(\frac{\sinh\sqrt{x}}{3}\right)$
   j) $f(x) = \cosh 3x\tanh 2x$
   k) $y = x\tanh^{-1}(\sin x)$

8. Find an equation for the tangent line to each of the following curves at each given point.
   a) $y = x\ln x,\ (e, e)$
   b) $f(x) = \sin^{-1} x,\ \left(\frac{\sqrt{2}}{2}, \frac{\pi}{4}\right)$

9. Find an equation for the tangent line at the given point.
   a) $\cos(xy^2) = y\ ,\ (0, 1)$
   b) $x^2 = \frac{x+y}{x-y}\ ,\ (1, 0)$

10. The position of a particle is given by the equation $s(t) = t^3 - 3t + 5$ (where $t \geq 0$ is measured in seconds and s is measured in meters).
    a) What is the velocity after 2 seconds?
    b) When is the particle at rest?
    c) When is the particle moving in the positive direction?
    d) Sketch a diagram to represent the motion of the particle.
    e) Find the total distance traveled during the first 3 seconds.
    f) Find the acceleration at time t and after 5 seconds.

11. How fast is the volume of a cylinder changing with respect to the radius when the radius is 4 mm and the height is a constant 5 mm?

12. The mass of a thin rod from the left end to a point x inches to the right is $x\left(1+\sqrt{x}\right)$ grams. Find the linear density when x is 4 inches.

13. A 5-meter ladder is leaning against a vertical wall. If the bottom of the ladder is pulled away from the wall at a rate of 2 meter/sec, at what rate is the top of the ladder moving down the wall when the top is 3 meters from the ground?

14. At noon, Alice is 4 miles south of Bob. Alice is running west at 5 mph and Bob walking east at 2 mph. How fast is the distance between Alice and Bob changing at 3pm?

15. Water runs into a conical cup at the rate of 2 in^3/min. The cup has a height of 4 inches and a base radius of 1 inch. How fast is the water level rising when the water is 3 inches deep?

16. Find the absolute maximum and minimum values of $f(x) = x\sqrt{1-x^2}$ on the interval $[-1,1]$.

17. Find the absolute maximum and minimum values of $f(x) = e^{\cos x}$ on the interval $[-\pi, 2\pi]$.

18. Verify that $f(x) = \sin x \cos x$ satisfies the three hypotheses of Rolle's Theorem on $\left[\frac{\pi}{3}, \frac{4\pi}{3}\right]$. Then find all numbers c that satisfy the conclusion of Rolle's Theorem.

19. Show that the equation $x^3 + \frac{4}{x^2} + 7 = 0$ has exactly one real solution on the interval $(-\infty, 0)$.

20. Verify that $f(x) = x \ln x$ satisfies the hypotheses of the Mean Value Theorem on $[1, e]$. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

21. Let $f(x) = \frac{1}{x^2-x-2} + 1$.
    a) Find the domain of f.
    b) Find the x-intercept(s) and y-intercept of f (if any).
    c) Find vertical asymptote(s) and horizontal asymptote(s) (if any).
    d) Find f' and f'', and determine where each are 0 and/or do not exist (DNE).
    e) Do a sign analysis on f' and f''.
    f) Find the intervals on which f is increasing and decreasing.
    g) Find the intervals on which f is concave up and concave down.
    h) Find all local maxima, local minima, and inflection points of f.
    i) Sketch the graph of f.

22. Let $f(x) = \frac{2x}{\sqrt{x^2+2}}$.
    a) Find the domain of f.
    b) Find the x-intercept(s) and y-intercept of f (if any).
    c) Find vertical asymptote(s) and horizontal asymptote(s) (if any).
    d) Find f' and f'', and determine where each are 0 and/or do not exist (DNE).
    e) Do a sign analysis on f' and f''.
    f) Find the intervals on which f is increasing and decreasing.
    g) Find the intervals on which f is concave up and concave down.
    h) Find all local maxima, local minima, and inflection points of f.
    i) Sketch the graph of f.

23. Let $f(x) = 3e^x(1-x)$.
    a) Find the domain of f.
    b) Find the x-intercept(s) and y-intercept of f (if any).
    c) Find vertical asymptote(s) and horizontal asymptote(s) (if any).
    d) Find f' and f'', and determine where each are 0 and/or do not exist (DNE).
    e) Do a sign analysis on f' and f''.
    f) Find the intervals on which f is increasing and decreasing.
    g) Find the intervals on which f is concave up and concave down.
    h) Find all local maxima, local minima, and inflection points of f.
    i) Sketch the graph of f.

24. A rectangle has its two lower corners on the x-axis and its two upper corners on the curve $y = 16 - x^4$. What is the maximum area of such a rectangle?

25. A right circular cylinder is inscribed in a sphere of radius r. Find the largest possible volume of such a cylinder.

26. A piece of wire 5 cm long is cut into two pieces. One piece is bent into a square and the other is bent into a circle. How should the wire be cut so that the total area enclosed is a minimum?

27. Use Newton's method to estimate the negative root of $x^4 + x - 3$ correct to six decimal places. Start with $x_1 = -1.5$.

INTEGRALS

28. A particle is moving with the given data. Find the position of the particle.
    a) $v(t) = 2t - \frac{1}{1+t^2}$, $s(0) = 1$
    b) $a(t) = \sin t + 3\cos t,\ s(0) = 0,\ v(0) = 2$

29. Find f if $f''(x) = 2x^3 + 3x^2 - 4x + 5$, $f(0) = 2$, and $f(1) = 0$.

30. Estimate the area under the graph of $f(x) = \ln x$ between $x = 1$ and $x = 3$ using…
    a) …a lower sum with four rectangles of equal width.
    b) …an upper sum with four rectangles of equal width.
    c) …midpoints with four rectangles of equal width.

31. Water is leaking out of a tank at $r(t)$ gallons per hour. Estimate the total amount of water that leaked out over 2 hours given the sample rates…

t (hours)	0	0.5	1	1.5	2
$r(t)$ (gallons per hour)	2.3	2.1	2.5	1.8	1.2

    a) …using left-endpoint values.
    b) …using right-endpoint values.

32. Evaluate the following integrals using Riemann sums with right endpoints.
    a) $\int_0^5 (x^2 - 2)\,dx$
    b) $\int_0^2 (x^3 + 3x)\,dx$

33. Suppose that $\int_{-2}^{3} f(x)\,dx = 5$ and $\int_{-2}^{-5} f(x)\,dx = 2$. Find the following.
    a) $\int_3^{-2} 4f(x)\,dx$
    b) $\int_{-5}^{3} 2f(x)\,dx$

34. Graph the following integrands and use the area under the graph to evaluate the integral.
    a) $\int_{-3}^{3} \left(1 + \sqrt{9 - x^2}\right) dx$
    b) $\int_{-5}^{0} (2 + |x + 3|)\,dx$

35. Use Part 1 of the Fundamental Theorem of Calculus to find the derivative of the following functions.
    a) $f(x) = \int_{x^2}^{2} \sqrt{t + 3}\,dt$
    b) $g(x) = \int_{2x}^{x+1} \frac{\sin t}{e^t}\,dt$

36. Evaluate the following integrals.
    a) $\int \left(\frac{2}{x^4} + \frac{3}{x\sqrt{x}} - \frac{1}{4x} + 5\sqrt[4]{x}\right) dx$

b) $\int \left(\cos\left(\frac{1}{3}x\right) + 3\csc x \cot x - e^{-2x+1} + 2^x\right) dx$

c) $\int \left(\frac{3}{2(1+x^2)} + \frac{1}{5\sqrt{1-x^2}} + \sec^2 x\right) dx$

d) $\int \frac{\sqrt{x}-3x^4}{x} dx$

e) $\int_1^2 \left(\frac{4}{x} - \frac{2x}{3}\right) dx$

f) $\int_{\pi/4}^{\pi/3} \csc x \cot x \, dx$

g) $\int_6^8 e^{\frac{1}{2}x-3} \, dx$

h) $\int_0^2 x^2\sqrt{1+x^3} \, dx$

i) $\int \frac{e^{\sqrt{x+2}}}{\sqrt{x+2}} dx$

j) $\int \sin^4 \pi x \cos \pi x \, dx$

k) $\int_{-\pi/4}^{\pi/4} \frac{x^2 \tan x}{3+\cos x} dx$

l) $\int_0^{\pi/8} \frac{\sec 2x \tan 2x}{3+\sec 2x} dx$

m) $\int \frac{\sin(\ln x)}{2x} dx$

n) $\int x^2 \cos 2x \, dx$

o) $\int e^{-x} \sin 3x \, dx$

p) $\int_0^1 x\sqrt{1-x} \, dx$

q) $\int x \sec^2 x \, dx$

r) $\int 2x \ln(x^2+1) \, dx$

37. Suppose the velocity function of a particle is $v(t) = 5\cos t$ (in meters per second). Find the displacement of the particle during the time period $0 \le t \le 2\pi$. Then find the distance traveled by the particle during the same time period $0 \le t \le 2\pi$.