L'Hospital's Rule

(covers Stewart 4.4)

Let's revisit limits for a moment. Consider the following limit:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

As $x \to 1$, the top$\to 0$ and the bottom$\to 0$. Limits of the type $\frac{0}{0}$ are said to be ____________________.

Previously, we would try to cancel a factor and then take the limit again. Now let's look at another powerful tool for evaluating such indeterminate limits.

L'Hospital's Rule

Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Ex 1.

Use L'Hospital's Rule to find the following limits.

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

$$\lim_{x \to 0} \frac{x - \sin x}{x^3}$$

$$\lim_{x \to 0} \frac{1 - \cos x}{x + x^2}$$

Note: L'Hospital's Rule also applies to one-sided limits as well (like $\lim_{x \to 0^+} \frac{\sin x}{x^2}$).

Type $\frac{\infty}{\infty}$

L'Hospital's Rule also works for indeterminate forms of type $\frac{\infty}{\infty}$.

Ex 2.

Find the following limits.

$$\lim_{x\to\infty} \frac{x^2}{e^{3x}}$$

$$\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}}$$

Type $\infty \cdot \mathbf{0}$ and $\infty - \infty$

Sometimes we can use algebra to convert limits of type $\infty \cdot 0$ or $\infty - \infty$ into type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ (then use L'Hospital's Rule).

Ex 3.

Find the following limits.

$$\lim_{x\to\infty} \left(x \sin\frac{1}{x}\right)$$

$$\lim_{x\to 0^+}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$$

Type $\mathbf{1}^{\infty}$, $\mathbf{0}^{\mathbf{0}}$, and $\infty^{\mathbf{0}}$

For these, try taking the logarithm of the function first, then take the limit, then exponentiate.

Ex 4.

Find the following limits.

$\lim_{x\to 0^+}(1+x)^{1/x}$

Derivatives of $\sin x$ and $\cos x$

As awesome as L'Hospital is, it can lead to circular reasoning. For example, what is the derivative of $\cos x$ using the definition of the derivative?

$$\frac{d}{dx}(\cos x) = \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} \underbrace{=}_{\text{L'Hospital}} \lim_{h\to 0}\frac{-\sin(x+h)}{1} = -\frac{\sin(x+0)}{1} = -\sin x$$

Why is this circular reasoning? Because to use L'Hospital, you need to know the derivative of $\cos x$. But that's what we're trying to find here! D'oh!

That's why we had to use the techniques we did when we proved the derivatives of $\sin x$ and $\cos x$.