Formula Derivations

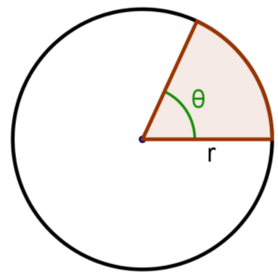
Math 160

Packet #19

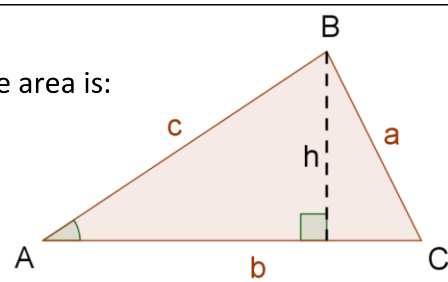
If θ is a central angle of a circle of radius r, and θ intercepts an arc of length s, then $\boldsymbol{\theta = \frac{s}{r}}$ is defined to be the **radian measure of $\boldsymbol{\theta}$**.

The formula $\theta = \frac{s}{r}$ also gives us a formula for **arc length**: $\boldsymbol{s = \theta r}$. (Note that θ is in radians.)

The **area of a sector** is: $\frac{\theta}{2\pi} \cdot \pi r^2 = \boldsymbol{\frac{1}{2} r^2 \theta}$
Here, θ is in radians and r is the radius of the circle.



Proof that $\boldsymbol{Area = \frac{1}{2} bc \sin A}$:
We can drop a perpendicular down to the base b (see right). So, the area is:
$Area = \frac{1}{2} bh$
But $\sin A = \frac{h}{c}$ so that $h = c \sin A$. Replacing h with $c \sin A$, we get:
$Area = \frac{1}{2} bc \sin A$
QED



Packet #21

$\boldsymbol{\sin^2 x + \cos^2 x = 1}$
Visual Proof: Draw a unit circle with a terminal point $(\cos t, \sin t)$. Draw a triangle from that point. The legs have sides $\cos t$ and $\sin t$, and the hypotenuse has length 1. The Pythagorean Theorem gives us $\cos^2 t + \sin^2 t = 1$. This is the same as the above identity.

$\boldsymbol{\tan^2 x + 1 = \sec^2 x}$
Proof: Divide both sides of $\sin^2 x + \cos^2 x = 1$ by $\cos^2 x$.

$\boldsymbol{1 + \cot^2 x = \csc^2 x}$
Proof: Divide both sides of $\sin^2 x + \cos^2 x = 1$ by $\sin^2 x$.

$\mathbf{\sin 2x = 2 \sin x \cos x}$

Proof: $\sin 2x = \sin(x + x) = \sin x \cos y + \cos x \sin x = 2 \sin x \cos x$

$\mathbf{\cos 2x = \cos^2 x - \sin^2 x}$

Proof: $\cos 2x = \cos(x + x) = \cos x \cos x - \sin x \sin x = \cos^2 x - \sin^2 x$

$\mathbf{\cos 2x = 1 - 2 \sin^2 x}$

Proof: $\cos 2x = \cos^2 x - \sin^2 x = (1 - \sin^2 x) - \sin^2 x = 1 - 2 \sin^2 x$

$\mathbf{\cos 2x = 2 \cos^2 x - 1}$

Proof: $\cos 2x = \cos^2 x - \sin^2 x = \cos^2 x - (1 - \cos^2 x) = 2 \cos^2 x - 1$

$\mathbf{\sin^2 x = \frac{1-\cos 2x}{2}}$

Proof:

$\cos 2x = 1 - 2 \sin^2 x$

$2 \sin^2 x = 1 - \cos 2x$

$\sin^2 x = \frac{1-\cos 2x}{2}$

$\mathbf{\cos^2 x = \frac{1+\cos 2x}{2}}$

Proof:

$\cos 2x = 2 \cos^2 x - 1$

$2 \cos^2 x = 1 + \cos 2x$

$\cos^2 x = \frac{1+\cos 2x}{2}$

$\mathbf{\tan^2 x = \frac{1-\cos 2x}{1+\cos 2x}}$

Proof: $\tan^2 x = \frac{\sin^2 x}{\cos^2 x} = \frac{\frac{1-\cos 2x}{2}}{\frac{1+\cos 2x}{2}} = \frac{1-\cos 2x}{2} \cdot \frac{2}{1+\cos 2x} = \frac{1-\cos 2x}{1+\cos 2x}$

$\mathbf{\sin \frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}}$

Proof:

$\sin^2 t = \frac{1-\cos 2t}{2}$

Replace t with $\frac{x}{2}$: $\sin^2 \frac{x}{2} = \frac{1-\cos 2\left(\frac{x}{2}\right)}{2}$

Square roots on both sides: $\sin \frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$

$\boldsymbol{\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}}$

Proof:

$\cos^2 t = \frac{1+\cos 2t}{2}$

Replace t with $\frac{x}{2}$: $\cos^2\frac{x}{2} = \frac{1+\cos 2\left(\frac{x}{2}\right)}{2}$

Square roots on both sides: $\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$