Binomial Theorem

(covers Sullivan 12.5)

How can we find $(a+b)^n$ for any natural number n? Let's try $n = 1, 2, 3, 4, \ldots$ and find patterns.

$(a+b)^1 = a + b$

$(a+b)^2 = a^2 + 2ab + b^2$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

Notice the exponent pattern (a goes from n to 0, and b goes from 0 to n).
The coefficient pattern is what's known as **Pascal's triangle**:

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

Ex 1.
Find the expansion of $(a+b)^7$ using Pascal's triangle.

Ex 2.
Find the expansion of $(2-3x)^5$ using Pascal's triangle.

To figure out the 15th row of Pascal's triangle, you first need to figure out rows 1 through 14.
When expanding $(a+b)^n$ for a large n, there's a more direct way to get the coefficients. But first…

Definitions:

$n! = n \cdot (n-1) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$ ($n!$ is read "***n* factorial**")

$0! = 1$

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$ ($\binom{n}{k}$ is called the **binomial coefficient**)

The Binomial Theorem:

$(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \cdots + \binom{n}{n-1} ab^{n-1} + \binom{n}{n} b^n$

(or, more compactly, $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k$)

Note: In combinatorics, $\binom{n}{k}$ counts the number of ways to choose k objects from n objects, where order doesn't matter. For example, the number of ways to choose a group of 4 students from a class of 36 students is $\binom{36}{4} = \frac{36!}{4!(36-4)!} = 58905$ ways.

For the Binomial Theorem, when you multiply out $(a+b)^n = (a+b)(a+b)(a+b) \cdots (a+b)$, each term is the product of a's and b's chosen from each factor. For example, if you don't choose any b's, then you're just multiplying all of the a's to get a^n, and there's only one way to do this. There are $\binom{n}{1} = n$ ways to choose one b and $n-1$ a's (that is, the term $a^{n-1}b$). In general, you can think of it like this: there are $\binom{n}{k}$ to choose k b's from the n factors of $a+b$.

Ex 3.

$\binom{9}{4} =$

$\binom{100}{3} =$

$\binom{100}{97} =$

Note: $\binom{n}{0} = 1$ $\binom{n}{1} = n$ $\binom{n}{n} = 1$

Ex 4.
Use the Binomial Theorem to expand $(2x-1)^4$.

Note: The term that contains a^{n-k} in the expansion of $(a+b)^n$ is: $\boldsymbol{\binom{n}{k} a^{n-k} b^k}$

Ex 5.
Find the coefficient of x^4 in the expansion of $(3x-2)^6$.

Note: The Binomial Theorem can be proved by induction! (see book for details)