Arithmetic and Geometric Sequences

(covers Sullivan 12.2 and 12.3)

Arithmetic Sequences

An **arithmetic sequence** is a sequence of the form: $a,\ a+d,\ a+2d,\ a+3d,\ \ldots$
a is the **first term**, and d is the **common difference**. The $\boldsymbol{n}$**th term** is given by $a_n =$ _______________

Ex 1.
Find the common difference and the 100th term of the following arithmetic sequence.
$11, 6, 1, -4, -9, \ldots$

Ex 2.
The 6th term of an arithmetic sequence is 10, and the 13th term is 31. Find the nth term and the 50th term.

Here's a slick way to find the sum of the first 100 numbers:
$S_{100} = 1 + 2 + 3 + 4 + \cdots + 97 + 98 + 99 + 100$
$S_{100} = 100 + 99 + 98 + 97 + \cdots + 4 + 3 + 2 + 1$

This works for all arithmetic sequences, so that the sum of the first n terms (S_n) is:

$$\boldsymbol{S_n = \frac{(a_1 + a_n)\cdot n}{2}}$$

Ex 3.
Find the sum: $3 + 7 + 11 + 15 + \cdots + 159$

Geometric Sequences

A **geometric sequence** is a sequence of the form: $a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ \ldots$
a is the **first term**, and r is the **common ratio**. The nth term is given by $a_n =$ __________

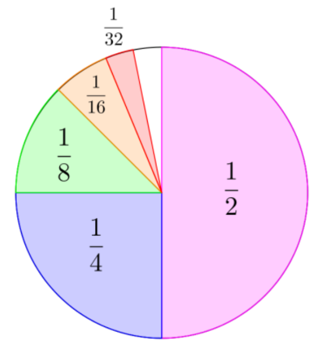
Ex 4.
Find the common ratio and the nth term of the following geometric sequence.
$2, -10, 50, -250, 1250, \ldots$

We can find the partial sums of a geometric sequence with another slick trick:
$S_n \quad = a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1}$
$rS_n = \qquad ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} + ar^n$
Subtract the equations to get:
$S_n - rS_n = a - ar^n$
$S_n(1 - r) = a(1 - r^n)$

$$\boldsymbol{S_n = \frac{a(1 - r^n)}{1 - r}}$$

Ex 5.
Find the sum: $4 - \frac{4}{3} + \frac{4}{9} - \frac{4}{27} + \cdots - \frac{4}{2187}$

Notice that the sum of the first n terms of the previous example is $S_n = \frac{4\cdot\left(1-\left(-\frac{1}{3}\right)^n\right)}{1-\left(-\frac{1}{3}\right)}$. What happens as we add more and more (and more!) terms? In other words, what happens as $n \to \infty$?



Well, $\left(-\frac{1}{3}\right)^n \to 0$, so $S_n \to \frac{4\cdot(1-0)}{1-\left(-\frac{1}{3}\right)} = 3$. So, the infinite number of terms add up to 3!

$4 - \frac{4}{3} + \frac{4}{9} - \frac{4}{27} + \cdots$ is an example of an **infinite series**.

To the right is a visual picture of the infinite geometric series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$

In general, if $-1 < r < 1$, then as $n \to \infty$, $r^n \to 0$, so $S_n = \frac{a(1-r^n)}{1-r} \to \frac{a}{1-r}$.

In this case, we say that the infinite series **converges** to $\frac{a}{1-r}$.

However, if $r \geq 1$ or $r \leq -1$, then the infinite series **diverges**.

Ex 6.

Determine whether the infinite geometric series converges or diverges. If it converges, find its sum.

$2 + \frac{2}{5} + \frac{2}{25} + \frac{2}{125} + \cdots$

Ex 7.

Determine whether the infinite geometric series converges or diverges. If it converges, find its sum.

$1 + \frac{7}{5} + \left(\frac{7}{5}\right)^2 + \left(\frac{7}{5}\right)^3 + \cdots$